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SINGLE ROTOR HELICOPTER PERFORMANCE ESTIMATION

A THESIS

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SINGLE ROTOR HELICOPTER PERFORMANCE ESTIMATION

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LIST OF SYMBOLS

a	Slope of lift curve for the blade element at 0.75 R (per radian)
a_o	Rotor coning angle
a_1	Lateral component of the cyclic pitch angle measured with respect to the tip-path plane
a_1^*	Reduced lateral cyclic pitch
A_o	Mean blade collective pitch angle positive above the tip-path plane
A_o^*	Reduced collective pitch angle
b	Number of blades in rotor
b_1	Longitudinal component of the cyclic pitch angle measured with respect to the tip-path plane
b_1^*	Reduced longitudinal cyclic pitch
c_{KN}	Coefficient to series representation of v_i where K and N are summation subscripts from K = 0 to N
c	Blade chord at non-dimensional radius x
$\bar{c}$	Mean blade chord
c_{d_o}	Section profile drag coefficient
c_l	Section lift coefficient
c_t	Blade chord at tip
c.t.	Refers to counter torque rotor
C_{M_o}	Mean blade root thrust moment coefficient
C_{M_x}	Mean rotor air rolling moment coefficient
C_{M_y}	Mean rotor air pitching moment coefficient
C_Q	Rotor torque coefficient
$\Delta\ C_{Q_{ao}}$	Increment to C_Q from term involving a_o
$\Delta\ C_{Q_{fn}}$	Nth increment to C_Q from fundamental terms involving C_Q

$\Delta C_{Q_{in}}$	Nth increment to C_Q from induced terms involving C_Q
ΔC_{Q_s}	Increment to C_Q from tip stall
C_T	Rotor thrust coefficient
C_x	Rotor X-force coefficient
C_{xy}	Rotor tangential-force coefficient, positive in direction of rotation
$\Delta C_{x_{fn}}$	Nth increment to C_x from fundamental terms involving C_x
$\Delta C_{x_{in}}$	Nth increment to C_x from induced terms involving C_x
C_y	Rotor y-force coefficient
C_z	Blade thrust coefficient
D_F	Parasite drag
f	As a subscript indicates "fundamental" components
F_x	Component of rotor resultant force acting along X-axis
F_y	Component of rotor resultant force acting along Y-axis
F_z	Component of rotor resultant force acting along Z-axis
HP	Horsepower
I_1	Mass moment of inertia of blade about flapping hinge
ℓ	Perpendicular distance from the main rotor shaft to the counter-torque rotor hub
L_F	Fuselage lift
m_o	The elemental blade root thrust moment for a single blade
m_x	Rolling moment for a single blade
m_y	Pitching moment for a single blade
M_o	Total mean blade root thrust moment for the rotor
M_x	Rotor rolling moment
M_y	Rotor pitching moment
Q	Rotor torque, negative in direction of rotation
r	Radius of blade element c dr

R	Radius of blade tip
s_{KN}	Coefficient to series representation of V_i where K and N are summation subscripts from $K = 0$ to N
s	As a subscript refers to stall
t	Maximum blade thickness at radius r
T	Rotor thrust, component of rotor resultant force along Z-axis
U	Component of resultant velocity at blade element that is normal to blade axis
v_a	Axial velocity ratio $v_a = \frac{V \sin \alpha_V}{\Omega R}$
v_a^*	Reduced normal velocity ratio (non-dimensional)
V	Freestream velocity
$V_{i(x,\Psi)}$	Normal component of induced velocity at blade axis
W	Gross weight of helicopter
x	Non-dimensional blade radius (r)
x_1	Non-dimensional distance from rotor hub to blade root
x_i	Non-dimensional effective radius $\approx .97$
$\bar{\alpha}$	Mean blade angle of attack in hovering
α_R	Blade element angle of attack measured from line of zero lift
α_V	Angle of attack of tip-path plane measured in XZ-plane between flight path velocity vector and tip-path plane, positive below tip-path plane
Γ	Circulation of blade element at radius r and azmuth angle
δ_o	Value of c_{d_o} at $c_l = 0$
ε	Constant for second term of even power series for c_{d_o}
η	Efficiency of power and drive system and to account for other miscellaneous losses
θ	Pitch angle of blade element at radius r and azmuth angle given by $\theta = A_o + \theta_1(3/4 - x) - a_1 \sin \Psi + b_1 \cos \Psi$ positive above tip-path plane

θ_1	Blade twist angle (change in angle of zero lift between blade stations $r = 0$ and $r = R$) positive for decreased angle at tip
θ_y	Angular displacement of tip-path plane about y-axis from horizontal
$\Delta\theta_t$	Blade twist from $x = 0.75$ to $x = 1$
μ	Inplane velocity ratio at tip-path plane $= \frac{V \cos \alpha_{\gamma}}{\Omega R}$
ρ	Density of air
σ	Rotor solidity
σ_t	Rotor solidity based on tip chord
σ_n	Constants which express blade-chord distribution
φ	Inflow angle at blade element
φ_c	Angle between flight path and horizontal position, below horizontal
Ψ	Azmuth angle of blade axis measured about Z-axis from X-axis
Ω	Mean angular velocity of rotor blade about Z-axis

SUMMARY

Equations are derived for determining the performance of single rotor helicopters. Charts were constructed to give solutions to the fundamental parts of the theoretical equations. The combined use of these charts and remaining equations provide a relatively rapid estimation of rotor performance for various flight conditions.

The equations are set up with no assumptions as to induced velocity distributions, which are represented within arbitrary limits of accuracy by a finite trigonometric series in rotor azmuth angle having polynomial coefficients in terms of the nondimensional blade radius. However, in the final performance equations, it was necessary to approximate the induced velocity distributions by the first few terms of the series, since the values for the coefficients of the other terms are presently unknown. The blade-element profile drag coefficient is represented by the first two terms of an even power series in the blade-element lift coefficient. Separate terms are developed for tip stall. Effects of compressibility on the blades are neglected. Expressions for the effect of blade twist are included in the basic equations.

Charts relating certain variables with functions from the performance equations were constructed in order to reduce the time and labor involved in the calculation of certain equations. However, charts were not constructed for those parts of the equations involving induced velocity components for which existing information is inadequate. The charts cover ranges of the in-plane velocity ratio from 0 through .4,

the thrust coefficient from 0.002 to 0.02, the mean blade element angle of attack in hovering from 0.05 to 0.14, and the axial velocity ratio from -0.10 to 0.02. It is felt that these values will cover current and future helicopter performance. Points for plotting the curves for the charts were determined utilizing an IBM 650 computer. The charts in conjunction with the equations afford a method for estimation of the rotor horsepower required by a helicopter flying at a given airspeed and at a given rate of climb or descent. A method of applying the charts and equations is given in an illustrated sample problem. It is also suggested that an agency which has a computer available could utilize the final equations given at the end of the analysis section before most of the approximations were introduced by programming a library routine to fit their particular needs.

A comparison of the results given by the performance equations with the full scale helicopter test data of NACA TN 1266 shows good agreement.

CHAPTER I

INTRODUCTION

Many methods for computing helicopter rotor performance have been presented based upon various simplifying assumptions. Most of these simplifying assumptions concern the distribution of induced velocity, which has been represented from an assumed uniform distribution to one of triangular distribution.

It is the purpose of this paper to develop a performance method based not upon any assumed induced velocity distribution, but based rather on the induced velocity represented within arbitrary limits of accuracy by a finite trigonometric series in rotor azmuth angle having polynomial coefficients in terms of the nondimensional blade radius. The accuracy of this representation is limited only by the information available as to the value of the induced velocity coefficients. Since improvement in this "phase of the art" is to be expected, the performance equations are separated into fundamental and induced velocity components. Thus charts which are constructed involving only the fundamental components of the performance equations should be valid now as well as in the future. The charts which are constructed, plus the few remaining equations which must be solved, give a relatively rapid estimation of rotor performance for various flight conditions. This estimation affords results which appear to be in good agreement with available flight test data. Separate terms are developed for tip stall.

The development of expressions for the reduced coefficients of collective and cyclic pitch angles was based upon work first done by Castles and Durham (1)* which uses a finite trigonometric series for induced velocity distribution. Since this work has not been published at this writing, it is reproduced in part in the first part of the analysis with the permission of the authors.

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*Numbers in parenthesis refer to items in the bibliography.

CHAPTER II

ANALYSIS

A. The development of expressions for the reduced coefficients of the collective and cyclic pitch angles which follows in Part A of the Analysis is the work of Castles and Durham (1). This development is based on representation of the induced velocity distribution as a finite trigonometric series.

Expression for the Blade Circulation Distribution.--The basic assumptions upon which the derivation in this section depends are as follows:

1. The effects of higher than the first harmonic blade flapping motion and blade deflection on the blade-bound vortex distribution can be considered as perturbations of small order.

2. Two-dimensional steady-state airfoil theory applies.

Under these assumptions the local blade element lift coefficient c_l is given by:

$$c_l = a \sin (\theta + \phi) = a(\sin \theta \cos \phi + \cos \theta \sin \phi) \tag{1}$$

where a = slope of blade element lift curve.

ϕ = inflow angle at blade element. (Figure 1)

θ = pitch angle of the blade element at radius r and azimuth angle Ψ given by:

$$\theta = A_0 + \theta_1 (3/4 - x) - a_1 \sin \Psi + b_1 \cos \Psi \tag{2}$$

in which

A_o = collective pitch angle at $r = 3/4\,R$.

θ_1 = blade twist angle (change in angle of zero lift between blade stations $r = 0$ and $r = R$).

x = nondimensional radius r/R.

a_1 = lateral cyclic pitch coefficient.

b_1 = longitudinal cyclic pitch coefficient.

In the above definitions the effects of higher harmonic blade flapping and blade deflection have been neglected in view of the assumptions made above.

Also by convention all blade angles are measured between the zero lift chord line of the blade element and the plane of rotation (tip-path plane). The blade azimuth angle Ψ is measured from the downwind position.

Substitution from equation (1) allows the local blade circulation x, Ψ, to be written in the form:

$$\Gamma_{x,\Psi} = \frac{1}{2}\; Ucc_1 = \frac{1}{2}\; ac \left\{ \sin\theta(U\cos\varphi) + \cos\theta(U\sin\varphi) \right\} \quad (3)$$

where U = component of resultant velocity perpendicular to the blade axis at blade station (x, Ψ)

c = blade chord at nondimensional radius x.

However,

$$U\cos\varphi = R(x + \mu\sin\Psi) \quad (4)$$

$$U\sin\varphi = R\left\{ v_a - a_o\,\mu\cos\Psi - \frac{V_{i(x,\Psi)}}{\Omega R} \right\} \quad (5)$$

in which Ω = rotor angular velocity.

R = rotor radius.

a_o = blade coning angle.

$V_{i(x,\psi)}$ = normal component of induced velocity at blade axis.

μ = $\dfrac{V \cos \alpha_V}{\Omega R}$.

v_a = $\dfrac{V \sin \alpha_V}{\Omega R}$.

V = freestream velocity.

α_V = angle of attack of tip-path plane.

Under the assumed flight conditions, the blade pitch angle θ is small and the approximations

$$\sin \theta \approx 0 \qquad \text{and} \qquad \cos \theta \approx 1 \tag{6}$$

are applicable. Hence, substitution from equations (2), (4), and (5) into equation (3) after expansion and rearrangement, takes the form:

$$\begin{aligned}\frac{2\Gamma_{x,\psi}}{ac\,\Omega R} = \Big\{ & (v_a - \tfrac{1}{2} a_1\mu) + (A_o + \tfrac{3}{4}\theta_1)\,x - \theta_1 x^2 + \\ & \Big[(A_o + \tfrac{3}{4}\theta_1)\mu - (a_1 + \theta_1\mu)x\Big]\sin\psi + (b_1 x - \\ & a_o\,\mu)\cos\psi + \tfrac{1}{2} b_1\mu \sin 2\psi + \tfrac{1}{2} a_1\mu\cos 2\psi \Big\} - \frac{V_{i(x,\psi)}}{\Omega R}\end{aligned} \tag{7}$$

For a given flight condition (where μ and v_a are known) the principal unknown quantities in this expression are the blade angles A_o, a_o, a_1, and b_1 and the induced velocity $V_{i(x,\psi)}$.

The right member of equation (7) consists of two main parts. The terms set apart by braces constitute a "fundamental" contribution independent of induced velocity, while the last term depends solely on induced effects. Thus, the local blade circulation can be written:

$$\Gamma_{x,\psi} = (\Gamma_f)_{x,\psi} + (\Gamma_i)_{x,\psi} \tag{8}$$

where, from equation (7)

$$\frac{2(\Gamma_f)_{x,\Psi}}{ac\,\Omega R} = (v_a - \frac{1}{2}\ a_1\mu) + (A_o + \frac{3}{4}\ \theta_1)x - \theta_1 x^2 \qquad (9)$$
$$+ \left[(A_o + \frac{3}{4}\theta_1)\mu \ - (a_1 + \theta_1\mu)x \right] \sin \Psi$$
$$+ (b_1 x - a_o\mu) \cos \Psi \ + \frac{1}{2} b_1\mu \ \sin 2\Psi \ + \frac{1}{2} a_1\mu \cos 2\Psi$$

and

$$\frac{2(\Gamma_i)_{x,\Psi}}{ac\,\Omega R} = - \frac{V_{i(x,\Psi)}}{\Omega R} \qquad (10)$$

In steady-state flight the distribution of normal component of induced velocity along a reference blade axis is a continuous function of the nondimensional radius x and azimuth angle Ψ , in fact, it is harmonic in Ψ . Thus it can be approximated to within arbitrary limits of accuracy by a finite trigonometric series in Ψ having polynomial coefficients in x. Such an approximation has the form:

$$\frac{(V_i)_{x,\Psi}}{\Omega R} = \sum_{K=0}^{N} \left[s_K(x) \sin K\Psi \ + c_K(x) \cos K\Psi \right] \qquad (11)$$

$$c_o(x) = c_{oo} + c_{o1}x + \dots\dots + c_{on_o}x^{n_o} \qquad (12)$$
$$\vdots$$
$$c_K(x) = c_{Ko} + c_{K1}x + \dots\dots + c_{Kn_K}x^{n_K}$$
$$s_K(x) = s_{Ko} + s_{K1}x + \dots\dots + s_{Km_K}x^{m_K}$$

<u>Expression for the Mean Rotor Blade Thrust Coefficient</u>.--Consider a right-circular cylindrical x, Ψ , Z coordinate system with the origin fixed at the rotor hub and let the positive Z-axis coincide with the upper tip-path plane axis. Define F_z the thrust acting on a reference blade at azimuth angle Ψ . Then the differential thrust acting on a blade element at radius r is given by:

$$dF_z = \rho(U \cos \varphi)\, \Gamma_{x,\varphi}\; dr = \frac{1}{2}\, \rho R(U \cos \varphi)\; (2\Gamma_{x,\varphi})dx \tag{13}$$

$$= \frac{1}{2}\; \rho\, ac\, \Omega R^2 (U \cos \varphi)\; \frac{2\Gamma_{x,\varphi}\; dx}{ac\Omega\, R}$$

define

$$C_z = \frac{F_z}{\frac{1}{2}\, \rho \pi \Omega^2 R^4} \tag{14}$$

Taking equation (13) and substituting from equations (4) and (8) using equation (14), resolving into "fundamental" and "induced" components, further substituting from equation (9), and expanding yields for the fundamental component:

$$\left[\frac{d}{dx}\left(\frac{c_z}{a}\right)\right]_f = \frac{c}{\pi R}\Big\{\Big[\frac{1}{2}\; \mu^2\, (A_o + \frac{3}{4}\; \theta_1) + (v_a - \mu\, a_1 \tag{15}$$

$$- \frac{1}{2}\; \mu^2\, \theta_1)x + (A_o + \frac{3}{4}\; \theta_1)x^2 - \theta_1 x^3\Big]$$

$$+\Big[\mu\,(v_a - \frac{3}{4}\,\mu a_1) + 2\mu\,(A_o + \frac{3}{4}\,\theta_1)x - (a_1 + 2\,\mu\,\theta_1)x^2\Big] \sin\varphi$$

$$+ \Big[\frac{1}{4}\; \mu^2\; b_1 - \mu\, a_o x + b_1 x^2\Big]\sin \Psi \;+ \Big[\frac{1}{4}\,\mu^2 b_1 - \mu\, a_o x$$

$$+ b_1 x^2\Big]\cos \Psi \;+ \Big[- \frac{1}{2}\mu^2\; a_o + \mu\, b_1 x\Big]\sin 2\Psi$$

$$+ \Big[- \frac{1}{2}\,\mu^2(A_o + \frac{3}{4}\,\theta_1) + \mu\,(a_1 + \frac{1}{2}\mu\,\theta_1)x\Big]\cos 2\Psi$$

$$+ \Big[\frac{1}{4}\,\mu^2 a_1\Big]\sin 3\varphi \;+ \Big[- \frac{1}{4}\,\mu^2 b_1\Big]\cos 3\varphi\;\Big\}$$

and substituting from equations (10) and (11) yields for the induced component:

$$\left[\frac{d}{dx}\left(\frac{c_z}{a}\right)\right]_i = -\frac{c}{\pi R}\left(x + \mu \sin \Psi\right)\sum_{K=0}^{N}\left[s_K(x) \sin K\Psi + c_K(x) \cos Kx\right] \qquad (16)$$

which could be further expanded by using equations (12) but in the interest of brevity will be omitted. Also note from the expansion of equation (16) it follows that:

$$\frac{d}{dx}\left(\frac{c_z}{a}\right) = \left[\frac{d}{dx}\left(\frac{c_z}{a}\right)\right]_f + \left[\frac{d}{dx}\left(\frac{c_z}{a}\right)\right]_i \qquad (17)$$

where the fundamental and induced components are given by equation (15) and the expanded form of equation (16), respectively.

At this stage it is convenient to further modify the above equations by means of the following definitions:

$$v_a^* = v_a - c_{oo} = \text{reduced normal velocity ratio} \qquad (18)$$

$$A_o^* = A_o - c_{o1} = \text{reduced collective pitch} \qquad (19)$$

$$a_1^* = a_1 + s_{11} = \text{reduced lateral cyclic pitch} \qquad (20)$$

$$b_1^* = b_1 - c_{11} = \text{reduced longitudinal cyclic pitch} \qquad (21)$$

Introducing the "starred" quantities into the fundamental component of equation (17) and appropriately adjusting the induced component of equation (16) in expanded form, results in the expression:

$$\frac{d}{dx}\left(\frac{c_z}{a}\right) = \left[\frac{d}{dx}\left(\frac{c_z}{a}\right)\right]_f^* + \left[\frac{d}{dx}\left(\frac{c_z}{a}\right)\right]_i^* \qquad (22)$$

Wherein the modified fundamental component now contains the principal induced effects and the adjusted induced component is now assumed to behave as a small-order perturbation. Written out, the modified fundamental component becomes:

$$
\begin{aligned}
\left[\frac{d}{dx}\left(\frac{c_z}{a}\right)\right]_f^* = \frac{c}{\pi R} \Bigg\{ \Bigg[& \frac{1}{2}\mu^2\left(A_o^* + \frac{3}{4}\theta_1\right) + \left(v_a^* - \mu a_1^* - \frac{1}{2}\mu^2\theta_1\right)x \\
& + \left(A_o^* + \frac{3}{4}\theta_1\right)x^2 - \theta_1 x^3 \Bigg] + \Bigg[\left(\mu v_a^* - \frac{3}{4}\mu^2 a_1^*\right) \\
& + \left(2\mu A_o^* + \frac{3}{4}\mu\theta_1\right)x + \left(-a_1^* - 2\mu\theta_1\right)x^2 \Bigg] \sin\Psi \\
& + \Bigg[\frac{1}{4}\mu^2 b_1^* - \mu a_o x + b_1^* x^2\Bigg]\cos\Psi + \Bigg[-\frac{1}{2}\mu^2 a_o \\
& + \mu b_1^* x\Bigg]\sin 2\Psi + \Bigg[-\frac{1}{2}\mu^2\left(A_o^* + \frac{3}{4}\theta_1\right) + \Big(\mu a_1^* \\
& + \frac{1}{2}\mu^2\theta_1\Big)x\Bigg]\cos 2\Psi + \Bigg[\frac{1}{4}\mu^2 a_1^*\Bigg]\sin 3\Psi \\
& + \Bigg[-\frac{1}{4}\mu^2 b_1^*\Bigg]\cos 3\Psi
\end{aligned}
\qquad (23)
$$

and the adjusted induced component becomes:

$$
\begin{aligned}
\left[\frac{d}{dx}\left(\frac{c_z}{a}\right)\right]_i^* = \frac{c}{\pi R} \Bigg\{ \Bigg[& \left(\frac{1}{2}\mu^2 c_{o1} - \frac{1}{2}\mu s_{1o}\right) + \frac{1}{2}\mu s_{11}x - \frac{1}{2}\mu s_{12}x^2 \\
& - \frac{1}{2}\mu s_{12}x^2 - \left(c_{o2} + \frac{1}{2}\mu s_{13}\right)x^3 \ldots \Bigg] + \Bigg[\frac{3}{4}\mu^2 s_{11} \\
& - \left(s_{1o} - \mu c_{o1}\right)x - \mu c_{o2}x^2 - \left(s_{12} + \mu c_{o3}\right)x^3 \ldots \Bigg] \sin\Psi \\
& + \Bigg[\left(\frac{1}{4}\mu^2 c_{11} - \frac{1}{2}\mu s_{2o}\right) - \left(c_{1o} + \frac{1}{2}\mu s_{21}\right)x - \frac{1}{2}\mu s_{22}x^2 \\
& - \left(c_{13} + \frac{1}{2}\mu s_{23}\right)x^3 \ldots \Bigg] \cos\Psi + \text{higher harmonic terms} \Bigg\}
\end{aligned}
\qquad (24)
$$

Integration of equation (22) over the blade span yields an expression from which the blade thrust coefficient C_z can be obtained:

$$\frac{C_z}{a} = \frac{1}{a} \left(C_{zf} + C_{zi}\right) \tag{25}$$

wherein the fundamental component C_{zf} is given by:

$$\frac{C_{zf}}{a} = \int_o^1 \left[\frac{d}{dx}\left(\frac{C_z}{a}\right)\right]_f^* dx \tag{26}$$

and the induced component is:

$$\frac{C_{zi}}{a} = \int_o^1 \left[\frac{d}{dx}\left(\frac{C_z}{a}\right)\right]_i^* dx \tag{27}$$

Substitution of equation (23) into (26) and carrying out the integration gives

$$\frac{C_{zf}}{a} = \left[\frac{1}{2}\, \mu^2 \left(A_o^* + \frac{3}{4}\,\theta_1\right) \sigma_1 + \left(v_a^* - \mu a_1^* - \frac{1}{2}\,\mu^2\, \theta_1\right) \sigma_2 \right. \tag{28}$$

$$\left. + \left(A_o^* + \frac{3}{4}\theta_1\right) \sigma_3 - \theta_1\, \sigma_4\right] + \left[\text{higher harmonic terms in } \Psi\right]$$

where

$$\sigma_n = \frac{1}{\pi R} \int_o^1 c\; x^{n-1}\, dx \tag{29}$$

Similarly, putting equation (24) into (27) yields:

$$\frac{C_{zi}}{a} = \left[\left(\frac{1}{2}\, \mu^2\, c_{o1} - \frac{1}{2}\mu\, s_{1o}\right) \sigma_1 + \frac{1}{2}\mu\, s_{11}\, \sigma_2 - \frac{1}{2}\mu\, s_{12}\, \sigma_3 \right. \tag{30}$$

$$\left. - \left(c_{o2} + \frac{1}{2}\,\mu s_{13}\right) \sigma_4 - \;\ldots\ldots\right] + \left[\text{higher harmonic}\right.$$

$$\left.\text{series in } \Psi\right]$$

The mean rotor blade thrust coefficient is defined by the relation:

$$C_T = \frac{T}{\rho \pi \Omega^2 R^4} \tag{31}$$

which can be evaluated by averaging the thrust force F_z for a single blade over one revolution in Ψ and multiplying by the number of blades b. Thus:

$$C_T = \frac{b}{2\pi} \int_0^{2\pi} \frac{F_z}{\rho \pi \Omega^2 R^4} \, d\varphi \tag{32}$$

Using equations (14) and (25), equation (32) can be written:

$$C_T = \frac{ab}{4\pi} \int_0^{2\pi} \frac{C_z}{a} \, d\Psi = \frac{ab}{4\pi} \int_0^{2\pi} \frac{1}{a} (C_{zf} + C_{zi}) d\Psi = C_{T_f} + C_{T_i} \tag{33}$$

wherein

$$C_{T_f} = \frac{ab}{4\pi} \int_0^{2\pi} \frac{C_{zf}}{a} \, d\Psi \qquad C_{T_i} = \frac{ab}{4\pi} \int_0^{2\pi} \frac{C_{zi}}{a} \, d\Psi \tag{34}$$

Substituting equations (28) and (30) into equations (34), and noting that the harmonic terms in Ψ vanish upon integration, gives:

$$\frac{2C_{T_f}}{ab\,\sigma_3} = \frac{1}{2}\mu^2 \left(A_o^* + \frac{3}{4}\theta_1\right) \frac{\sigma_1}{\sigma_3} + \left(v_a^* - \mu a_1^* - \frac{1}{2}\mu^2 \theta_1\right) \frac{\sigma_2}{\sigma_3} + \left(A_o^* + \frac{3}{4}\theta_1\right) - \theta_1 \frac{\sigma_4}{\sigma_3} \tag{35}$$

and

$$\frac{2C_{T_i}}{ab\,\sigma_3} = \left(\frac{1}{2}\mu^2 c_{01} - \frac{1}{2}\mu s_{10}\right) \frac{\sigma_1}{\sigma_3} + \frac{1}{2}\mu s_{11} \frac{\sigma_2}{\sigma_3} - \frac{1}{2}\mu s_{12} - \left(c_{12} + \frac{1}{2}\mu s_{13}\right) \frac{\sigma_4}{\sigma_3} \;\ldots. \tag{36}$$

For constant chord blades:

$$\sigma_n = \frac{1}{\pi R} \int_0^1 cx^{n-1}\,dx = \frac{c}{\pi R n} \qquad n \geq 1 \tag{37}$$

In this case equation (35) reduces to:

$$\frac{2C_{T_f}}{ab\,\sigma_3} = (1 + \frac{3}{2}\mu^2)A_o^* + \frac{3}{8}\mu^2\,\theta_1 + \frac{3}{2}v_a^* - \frac{3}{2}\mu a_1^* \tag{38}$$

and equation (36):

$$\frac{2C_{T_i}}{ab\,\sigma_3} = (\frac{3}{2}\mu^2\,c_{o1} - \frac{3}{2}\,\mu s_{1o}) + \frac{3}{4}\,\mu s_{11} - \frac{1}{2}\,\mu s_{12}$$

$$- (\frac{3}{4}\,c_{o2} + \frac{3}{8}\,\mu\,s_{13}) - \ldots \tag{39}$$

If it be assumed, at least temporarily, that the induced contribution (39) is small compared to the fundamental contribution (38), then as an approximation:

$$\frac{2C_T}{ab\,\sigma_3} \approx \frac{2C_{T_f}}{ab\,\sigma_3} = (1 + \frac{3}{2}\mu^2)\,A_o^* + \frac{3}{8}\,\mu^2\,\theta_1 + \frac{3}{2}\,v_a^* - \frac{3}{2}\mu\,a_1^* \tag{40}$$

for rotors having constant chord blades.

<u>Expression for the Mean Rotor Air Rolling Moment Coefficient</u>.--The rolling moment for a blade element at nondimensional radius x and azimuth angle φ can be written

$$dm_x = x\sin\Psi\;\;dF_z = \frac{1}{2}\;a\,\rho\,\pi\,\Omega^2R^5\;x\sin\varphi\;\frac{d}{dx}\left(\frac{C_z}{a}\right)dx \tag{41}$$

Thus, the rolling moment for a single blade is

$$m_x = \frac{1}{2} a \rho \pi \Omega^2 R^5 \sin \Psi \int_0^1 x \frac{d}{dx}\left(\frac{C_z}{a}\right) dx \tag{42}$$

from which, using equation (22)

$$m_x = m_{x_f} + m_{x_i} \tag{43}$$

where

$$m_{x_f} = \frac{1}{2} a \rho \pi \Omega^2 R^5 \sin \Psi \int_0^1 x \left[\frac{d}{dx}\left(\frac{C_z}{a}\right)\right]_f^* dx \tag{44}$$

and

$$m_{x_i} = \frac{1}{2} a \rho \pi \Omega^2 R^5 \sin \Psi \int_0^1 x \left[\frac{d}{dx}\left(\frac{C_z}{a}\right)\right]_i^* dx \tag{45}$$

Putting equation (23) into (44) and integrating yields:

$$\begin{aligned} m_{x_f} = \frac{1}{2} a \rho \pi \Omega^2 R^5 \Bigg\{ & \Big[\frac{1}{2} \mu^2 (A_o^* + \frac{3}{4} \theta_1) \sigma_2 + (v_a^* - \mu a_1^* \\ & - \frac{1}{2} \mu^2 \theta_1) \sigma_3 + (A_o^* + \frac{3}{4} \theta_1) \sigma_4 - \theta_1 \sigma_5 \Big] \sin \Psi \\ & + \Big[(\mu v_a^* - \frac{3}{4} \mu^2 a_1^*) \sigma_2 + (2 \mu A_o^* + \frac{3}{2} \mu \theta_1) \sigma_3 \\ & + (-a_1^* - 2 \mu \theta_1) \sigma_4 \Big] \sin^2 \Psi + \Big[\text{products of higher} \\ & \text{harmonics}\Big] \end{aligned} \tag{46}$$

Similarly, putting equation (24) into (45) gives:

$$m_{x_i} = \frac{1}{2}\ a\,\rho\,\pi\,\Omega^2 R^5 \left\{\left[(\frac{1}{2}\mu^2\ c_{o1} - \frac{1}{2}\mu\,s_{1o})\ \sigma_2 + \frac{1}{2}\mu\,s_{11}\,\sigma_3 \right.\right. \tag{47}$$

$$\left. - \frac{1}{2}\mu\,s_{11}\ \sigma_4 - (c_{o2} + \frac{1}{2}\mu\,s_{13})\,\sigma_5 - \ldots\ldots\right] \sin\Psi$$

$$+ \left[\frac{3}{4}\mu^2 s_{11}\,\sigma_2 - (s_{1o} - \mu\,c_o)\,\sigma_3 - \mu\,c_{o2}\,\sigma_4\right.$$

$$\left. - (s_{12} + \mu\,c_{o3})\ \sigma_5 - \ldots\ldots\right]\ \sin^2\Psi$$

$$\left. + \left[\text{series of products of higher harmonics}\right]\right\}$$

Multiplication of equation (46) by the number of blades b and averaging over a revolution of Ψ yields the mean fundamental component of rotor air rolling moment:

$$M_{x_f} = \frac{1}{2}\ \rho\pi\Omega^2 R^5\ (\frac{ab}{2}\ )\ \left[(\mu\,v_a^* - \frac{3}{4}\mu^2 a_1^*)\,\sigma_2 + (2\mu\,A_o^*\right. \tag{48}$$

$$\left. + \frac{3}{2}\ \mu\,\theta_1)\,\sigma_3 - (a_1^* + 2\mu\,\theta_1)\,\sigma_4\right]$$

Similarly, for the induced component:

$$M_{x_i} = \frac{1}{2}\rho\pi\ \Omega^2 R^5\ (\frac{ab}{2})\ \left[\frac{3}{4}\mu^2 s_{11}\ \sigma_2 - (s_{1o} - \mu\,c_{o1})\,\sigma_3\right. \tag{49}$$

$$\left. - \mu\,c_{o2}\,\sigma_4 - (s_{12} + \mu\,c_{o3})\ \sigma_5 - \ldots\ldots\right]$$

If the mean rotor air rolling moment coefficients $C_{M_{xf}}$ and $C_{M_{xi}}$ be defined by the relations:

$$C_{M_{xf}} = \frac{M_{x_f}}{\frac{1}{2}\ \rho\,\pi\,\Omega^2 R^5} \tag{50}$$

$$C_{M_{xi}} = \frac{M_{x_i}}{\frac{1}{2}\ \rho\,\pi\,\Omega^2 R^5} \tag{51}$$

And the total mean air rolling moment C_{M_x} is defined by the relation:

$$C_{M_x} = \frac{M_x}{\frac{1}{2} \rho \pi \Omega^2 R^5} \tag{52}$$

where

$$M_x = M_{x_f} + M_{x_i} \tag{53}$$

and

$$\frac{2C_{m_x}}{ab\sigma_4} = \frac{2}{ab\sigma_4} (C_{M_{xf}} + C_{M_{xi}}) \tag{54}$$

Using the above relations in equations (48) and (49) and utilizing equations (36) and (37) it follows for constant chord blades:

$$\frac{2C_{M_{xf}}}{ab\sigma_4} = \frac{8}{3} \mu A_o^* + 2\mu v_a^* - (1 + \frac{3}{2} \mu^2)a_1^* \tag{55}$$

$$\frac{2C_{M_{xi}}}{ab\sigma_4} = (-\frac{3}{2} \mu^2)s_{11} - \frac{4}{3}(s_{1o} - \mu c_{o1}) - \mu c_{o2} - \frac{4}{5}(s_{12} + \mu c_{o3}) - \ldots \tag{56}$$

Hence, if it be temporarily assumed that the induced contribution (56) is negligible compared to (55), then as an approximation:

$$\frac{2C_{M_x}}{ab\sigma_4} \approx \frac{2C_{M_{xf}}}{ab\sigma_4} = \frac{8}{3} \mu A_o^* + 2\mu v_a^* - (1 + \frac{3}{2} \mu^2)a_1^* \tag{57}$$

for rotors having constant chord blades.

Expression for the Mean Rotor Air Pitching Moment Coefficient.--In a manner analogous to that of the preceding section, the elemental pitching moment is defined by the relation:

$$dm_y = x \cos \Psi \ \ dF_z = \frac{1}{2} a \rho \pi \Omega^2 R^5 x \cos \Psi \ \ \frac{d}{dx}\left(\frac{C_z}{a}\right) dx \qquad (58)$$

from which the pitching moment for a single blade is found to be

$$m_y = \frac{1}{2} a \rho \pi \Omega^2 R^5 \cos \Psi \int_0^1 x \frac{d}{dx}\left(\frac{C_z}{a}\right) dx = m_{y_f} + m_{y_i} \qquad (59)$$

where

$$m_{y_f} = \frac{1}{2} a \rho \pi \Omega^2 R^5 \cos \Psi \int_0^1 x \left[\frac{d}{dx}\left(\frac{C_z}{a}\right)\right]_f^* dx \qquad (60)$$

$$m_{y_i} = \frac{1}{2} a \rho \pi \Omega^2 R^5 \cos \Psi \int_0^1 x \left[\frac{d}{dx}\left(\frac{C_z}{a}\right)\right]_i^* dx \qquad (61)$$

Substituting equation (23) into (60) and anticipating that when the averaging with respect to Ψ is carried out at a later stage all terms will drop out except the $\cos^2 \Psi$ term, gives:

$$m_{y_f} = \frac{1}{2} a \rho \pi \Omega^2 R^5 \left[\frac{1}{4} \mu^2 b_1^* \sigma_2 - \mu a_o \sigma_3 + b_1^* \sigma_4\right] \cos^2 \Psi \qquad (62)$$

(other harmonic terms in Ψ)

Similarly, putting equation (24) into (61) gives:

$$m_{y_i} = \frac{1}{2} a \rho \pi \Omega^2 R^5 \left[\left(\frac{1}{4} \mu^2 c_{11} - \frac{1}{2} \mu s_{2o}\right) \sigma_2 - \left(c_{1o} + \frac{1}{2} \mu s_{21}\right)\sigma_3 - \frac{1}{2} \mu s_{22} \sigma_4 - \left(c_{13} + \frac{1}{2} \mu s_{23}\right) \sigma_5 - \ldots\right] \cos^2 \Psi + \qquad (63)$$

(series of harmonics in Ψ ......)

Multiplying equations (62) and (63) by the number of blades b and averaging over a revolution of Ψ then yields the component mean air pitching moments:

$$M_{y_f} = \frac{1}{2}\rho\pi\Omega^2 R^5 \left(\frac{ab}{2}\right)\left[\frac{1}{4}\mu^2 b_1^* \sigma_2 - \mu a_o \sigma_3 + b_1^* \sigma_4\right] \quad (64)$$

and

$$M_{y_i} = \frac{1}{2}\rho\pi\Omega^2 R^5 \left(\frac{ab}{2}\right)\left[\left(\frac{1}{4}\mu^2 c_{11} - \frac{1}{2}\mu s_{20}\right)\sigma_2 - \left(c_{1o} + \frac{1}{2}\mu s_{21}\right)\sigma_3 - \frac{1}{2}\mu s_{22}\sigma_4 - \left(c_{13} + \frac{1}{2}\mu s_{23}\right)\sigma_5 - \ldots\right] \quad (65)$$

The component mean air pitching moment coefficients are defined as:

$$C_{M_{yf}} = \frac{M_{y_f}}{\frac{1}{2}\rho\pi\Omega^2 R^5} \quad (66)$$

and

$$C_{M_{yi}} = \frac{M_{y_i}}{\frac{1}{2}\rho\pi\Omega^2 R^5} \quad (67)$$

And the total air pitching moment coefficient C_{M_y} is defined by:

$$C_{M_y} = \frac{M_y}{\frac{1}{2}\rho\pi\Omega^2 R^5} \quad (68)$$

where

$$M_y = M_{y_f} + M_{y_i} \quad (69)$$

and

$$\frac{2C_{M_y}}{ab\sigma_4} = \frac{2}{ab\sigma_4}\left(C_{M_{yf}} + C_{M_{yi}}\right) \quad (70)$$

Using the above relations in equations (64) and (65) and utilizing equations (36) and (37) it follows for constant chord blades:

$$\frac{2C_{M_{yf}}}{ab\sigma_4} = \left(1 + \frac{1}{2}\mu^2\right)b_1^* - \frac{4}{3}\mu a_o \quad (71)$$

$$\frac{2C_{M_{yi}}}{ab\sigma_4} = (\frac{1}{2}\mu^2 c_{11} - \mu s_{2o}) - \frac{4}{3}(c_{1o} + \frac{1}{2}\mu s_{21}) - \frac{1}{2}\mu s_{22} \cdots\cdots \quad (72)$$

The further assumption of negligible induced contribution then gives rise to the approximation

$$\frac{2C_{M_y}}{ab\sigma_4} \approx \frac{2C_{M_{yf}}}{ab\sigma_4} = (1 + \frac{1}{2}\mu^2)b_1^* - \frac{4}{3}\mu a_o \quad (73)$$

for rotors having constant chord blades.

Expression for the Mean Blade Root Thrust Moment Coefficient.--The elemental blade root thrust moment is given by:

$$dm_o = xdF_z = \frac{1}{2} a \rho \pi \Omega^2 R^5 x \frac{d}{dx}\left(\frac{C_z}{a}\right) dx \quad (74)$$

so that for a single blade the thrust moment is:

$$m_o = \frac{1}{2} a \rho \pi \Omega^2 R^5 \int_o^1 x \frac{d}{dx}\left(\frac{C_z}{a}\right) dx = m_{of} + m_{oi} \quad (75)$$

where

$$m_{of} = \frac{1}{2} a \rho \pi \Omega^2 R^5 \int_o^1 x \left[\frac{d}{dx}\left(\frac{C_z}{a}\right)\right]_f^* \quad (76)$$

$$m_{oi} = \frac{1}{2} a \rho \pi \Omega^2 R^5 \int_o^1 x \left[\frac{d}{dx}\left(\frac{C_z}{a}\right)\right]_i^* \quad (77)$$

Substituting equations (23) and (24) into (76) and (77), respectively; then averaging with respect to Ψ, gives the mean blade root thrust moment coefficients $C_{M_{of}}$ and $C_{M_{oi}}$ in the form:

$$\frac{C_{M_{of}}}{a\,\sigma_4} = \frac{M_{of}}{a\,\sigma_4\,\frac{1}{2}\,\rho\pi\,\Omega^2 R^5} = \frac{1}{2}\mu^2\left(A_o^* + \frac{3}{4}\theta_1\right)\frac{\sigma_2}{\sigma_4} + \left(v_a^* - \mu a_1^*\right. \quad (78)$$

$$\left. - \frac{1}{2}\mu^2\theta_1\right)\frac{\sigma_3}{\sigma_4} + \left(A_o^* + \frac{3}{4}\theta_1\right) - \theta_1\frac{\sigma_5}{\sigma_4}$$

$$\frac{C_{M_{oi}}}{a\,\sigma_4} = \frac{M_{oi}}{a\,\sigma_4\,\frac{1}{2}\,\rho\pi\,\Omega^2 R^5} = \left(\frac{1}{2}\mu^2 c_{o1} - \frac{1}{2}\mu s_{1o}\right)\frac{\sigma_2}{\sigma_4} \quad (79)$$

$$- \frac{3}{2}\mu s_{11}\frac{\sigma_3}{\sigma_4} + \frac{1}{2}\mu s_{11}\frac{\sigma_3}{\sigma_4} - \frac{1}{2}\mu s_{12} - \left(c_{o2} + \frac{1}{2}\mu s_{13}\right)\frac{\sigma_5}{\sigma_4} - \ldots.$$

Upon defining the total mean blade root thrust moment M_o by:

$$M_o = M_{o_f} + M_{o_i} \quad (80)$$

and the mean blade root thrust moment coefficient C_{M_o} by:

$$C_{M_o} = \frac{M_o}{\frac{1}{2}\rho\pi\Omega^2 R^5} \quad (81)$$

It follows that:

$$\frac{C_{M_o}}{ab\sigma_4} = \frac{1}{a\,\sigma_4}\left(C_{M_{of}} + C_{M_{oi}}\right) \quad (82)$$

Then for constant chord blades equations (78) and (79) reduce to

$$\frac{C_{M_{of}}}{a\,\sigma_4} = (1 + \mu^2)A_o^* - \frac{1}{20}\left(1 - \frac{5}{3}\mu^2\right)\theta_1 + \frac{4}{3}v_a^* - \frac{4}{3}\mu a_1^* \quad (83)$$

$$\frac{C_{M_{oi}}}{a\sigma_4} = \mu^2 c_{o1} - \mu s_{1o} + \frac{2}{3}\mu s_{11} - \frac{1}{2}\mu s_{12} - \frac{4}{5}\left(c_{o2} - \frac{1}{2}\mu s_{13}\right) \quad (84)$$

Where upon the assumption that the contribution of (84) is small leads to the approximation:

$$\frac{C_{M_o}}{a\sigma_4} \approx \frac{C_{M_{of}}}{a\,\sigma_4} = (1 + \mu^2)\,A_o^* - \frac{1}{20}\left(1 - \frac{5}{3}\mu^2\right)\theta_1 + \frac{4}{3}v_a^* - \frac{4}{3}\mu a_1^* \quad (85)$$

Approximations for Equilibrium Values of A_o^*, a_o, a_1^*, and b_1^*.--To develop performance equation, for steady-state flight, the rotor rolling and pitching moments may be assumed to be zero. Using this fact, equations (40), (57), (73), and (85) for constant chord blades give:

$$\left(1 + \frac{3}{2}\mu^2\right)A_o^* - \frac{3}{2}\mu a_1^* + \frac{3}{2}v_a^* + \frac{3}{8}\mu^2\theta_1 = \frac{2C_T}{ab\,\sigma_3} \quad (86)$$

$$\frac{8}{3}\mu A_o^* - \left(1 + \frac{3}{2}\mu^2\right)a_1^* + 2\mu v_a^* = 0 \quad (87)$$

$$\left(1 + \frac{1}{2}\mu^2\right)b_1^* - \frac{4}{3}\mu a_o = 0 \quad (88)$$

$$(1 + \mu^2)A_o^* - \frac{4}{3}\mu a_1^* + \frac{4}{3}v_a^* - \frac{1}{20}\left(1 - \frac{5}{3}\mu^2\right)\theta_1 = \frac{C_{M_o}}{a\,\sigma_4} \quad (89)$$

Solving the first pair of these equations for A_o^* and a_1^* yields:

$$A_o^* = \frac{\left(1 + \frac{3}{2}\mu^2\right)\bar{\alpha} - \frac{3}{8}\mu^2\left(1 + \frac{3}{2}\mu^2\right)\theta_1 - \frac{3}{2}\left(1 - \frac{1}{2}\mu^2\right)v_a^*}{\left(1 + \frac{3}{2}\mu^2\right)^2 - 4\mu^2} \quad (90)$$

$$a_1^* = \frac{\mu\left[\frac{8}{3}\ \bar{\alpha} - \mu^2\theta_1 - 2(1 - \frac{3}{2}\mu^2)\ v_a^*\right]}{(1 + \frac{3}{2}\mu^2)^2 - 4\mu^2} \tag{91}$$

in which

$$\bar{\alpha} = \frac{2C_T}{ab\,\sigma_3} \tag{92}$$

Equation (88) gives for b_1^*:

$$b_1^* = \frac{\frac{4}{3}\mu\,a_o}{(1 + \frac{1}{2}\mu^2)} \tag{93}$$

Substitution for A_o^* and a_1^* from equations (90) and (91) into (89) leads to:

$$\frac{C_{M_o}}{a\,\sigma_4} = \frac{1}{(1 - \mu^2 + \frac{9}{4}\mu^4)}\left[(1 - \frac{19}{18}\mu^2 + \frac{3}{2}\mu^4)\ \bar{\alpha} - \frac{1}{20}(1 + \frac{29}{6}\mu^2 - 4\mu^4 + \frac{33}{4}\mu^6)\ \theta_1 - \frac{1}{6}(1 - \frac{7}{2}\mu^2 + \frac{3}{2}\mu^4)v_a^*\right] \tag{94}$$

However,

$$C_{M_o} = \frac{M_o}{\frac{1}{2}\rho\pi\Omega^2R^5} \approx \frac{a_o\ \Omega^2 I_1}{\frac{1}{2}\rho\pi\Omega^2R^5} = \frac{2a_oI_1}{\rho\pi R^5} \tag{95}$$

where I_1 equals the moment of inertia of one rotor blade about the flapping hinge. Dividing C_{M_o} by $a\,\sigma_4$ and putting this approximation into equation (94) and solving for a_o gives:

$$a_o \approx \frac{a\sigma_4\,\rho\pi R^5}{2I_1(1 - \mu^2 + \frac{9}{4}\mu^4)}\left[(1 - \frac{19}{18}\mu^2 + \frac{3}{4}\mu^4)\ \bar{\alpha} - \frac{1}{20}(1 + \frac{29}{6}\mu^2 - 4\mu^4 + \frac{33}{4}\mu^6)\theta_1 - \frac{1}{6}(1 - \frac{7}{2}\mu^2 + \frac{3}{2}\mu^4)v_a^*\right] \tag{96}$$

Mean Blade Circulation.--If equation (9) is modified by means of equations (18) and (19) and the resulting correction terms absorbed into equation (10), the results are:

$$\frac{2(\Gamma_f)^*_{x,\Psi}}{ac\Omega R} = (v_a^* - \frac{1}{2}\mu a_1^*) + (A_o^* + \frac{3}{4}\theta_1)x - \theta_1 x^2 \tag{97}$$

$$+ \left[\mu(A_o^* + \frac{3}{4}\theta_1) - (a_1^* + \mu\theta_1)x\right] \sin\Psi$$

$$+ (b_1^* x - \mu a_o)\cos\Psi + \frac{1}{2}\mu b_1^* \sin 2\Psi$$

and

$$\frac{2(\Gamma_i)^*_{x,\Psi}}{ac\Omega R} = -\left[\frac{1}{2}\mu s_{11} + c_{o2}x^2 + c_{o3}x^3 + \ldots\right] - \left[(s_{1o} - \right. \tag{98}$$

$$\left. - \mu c_{o1}) + s_{12}x^2 + \ldots\right] \sin\Psi - \left[c_{1o} + c_{12}x^2\right.$$

$$\left. + c_{13}x^3 + \ldots\right]\cos\Psi - \left[(s_{20} - \frac{1}{2}\mu c_{11})\right.$$

$$\left. + s_{21}x + \ldots\right] \sin 2\Psi - \left[(c_{20} - \frac{1}{2}\mu s_{11} + c_{21}x\right.$$

$$\left. + c_{22}x^2 + \ldots\right] \cos 2\Psi - \text{(series of higher}$$

$$\text{harmonics in } \Psi)$$

B. Using the expressions for the reduced coefficients A_o^*, a_1^*, b_1^*, a_o given in Part A of the Analysis, Performance Equations are now developed.

Equation for the Rotor Blade XY-Force.--The inplane component of force ΔF_{xy} acting on a blade element at radius r and azimuth angle Ψ, acting chordwise, positive in the direction of rotation, (see Figure 2) may be expressed as:

$$\Delta F_{xy} = - \rho\Gamma (U \sin \varphi) \, dr + \frac{1}{2} \rho U (U \cos \varphi) \, c \, c_{d_o} \, dr \tag{99}$$

where c_{d_o} = blade element profile drag coefficient. The relationship between the profile drag coefficient c_{d_o} and the lift coefficient c_l of the blade element airfoil has been approximated by Castles (2) by the first two terms of an even power series in c_l as follows:

$$c_{d_o} = \delta_o + \varepsilon c_\ell^2 + \ldots\ldots \tag{100}$$

Then

$$\Delta F_{xy} = - \rho (U \sin \varphi) \, \Gamma \, dr + \frac{1}{2} \rho U (U \cos \varphi) c \, \delta_o dr + \frac{1}{2} \rho U (U \cos \varphi) \left(\varepsilon c_\ell^2 \, dr \right) \tag{101}$$

Define a coefficient

$$C_{xy} = \frac{F_{xy}}{\frac{1}{2} \rho \pi \Omega^2 R^4} \tag{102}$$

where F_{xy} is the total inplane component of force acting on the blade at azimuth angle Ψ, positive in the direction of rotation. Then writing

$$C_{xy} = (C_{xy})_a - (C_{xy})_{\delta_o} - (C_{xy})_\varepsilon \tag{103}$$

it follows from equation (102) that $(C_{xy})_a$, the part of the coefficient due to lift, is:

$$\frac{(C_{xy})_a}{a} = \frac{1}{\pi R} \int_{x_1}^{x_1} c \left(\frac{U \sin \varphi}{\Omega R} \right) \left(\frac{2 \Gamma}{a c \Omega R} \right) dx \tag{104}$$

and the part arising from δ_o, the constant term in the even power series (100) above for the profile drag coefficient, is:

$$\frac{(C_{xy})_{\delta_o}}{\delta_o} = \frac{1}{\pi R} \int_{x_1}^{x_i} c \left(\frac{U \cos \varphi}{\Omega R}\right)^2 \frac{1}{\cos \varphi} \, dx \qquad (105)$$

and the part associated with the variation of profile drag coefficient with lift coefficient is:

$$\frac{(C_{xy})_{\varepsilon}}{\varepsilon a^2} = \frac{1}{\pi R} \int_{x_1}^{x_i} c \left(\frac{U \cos \varphi}{\Omega R}\right)^2 \left(\frac{c_\ell}{a}\right)^2 \frac{1}{\cos \varphi} \, dx \qquad (106)$$

Noting that $\frac{c_\ell}{a} = 2\Gamma \frac{\cos \varphi}{ac(U \cos \varphi)}$, by rearrangement of equation (3), equation (106) may be rewritten:

$$\frac{(C_{xy})_{\varepsilon}}{\varepsilon a^2} = \frac{1}{\pi R} \int_{x_1}^{x_i} c \left(\frac{2\Gamma}{ac\,\Omega R}\right)^2 \cos \varphi \, dx \qquad (107)$$

Since $(C_{xy})_\varepsilon$ is a small term, the value of $\cos \varphi$ in equation (107) can be set equal to unity introducing a very small conservative error. The value of $\frac{1}{\cos \varphi}$ in equation (105) can be approximated as follows:

$$\frac{1}{\cos \varphi} = \sqrt{1 + \tan^2 \varphi}$$

and using the binomial expansion dropping small terms yields:

$$\frac{1}{\cos \varphi} \approx 1 + \frac{1}{2} \tan^2 \varphi \qquad (108)$$

Now dividing equation (5) by equation (4) gives:

$$\tan \varphi = \frac{v_a - a_o \mu \cos \varphi - \frac{v_{i(x,\Psi)}}{\Omega R}}{x + \mu \sin \Psi} \tag{109}$$

It now follows from substitution of the values from equations (108) and (109) into equation (105), considering $a_o \mu \cos \varphi$ a small term, that:

$$\frac{(C_{xy})_{\delta_o}}{\delta_o} = \frac{1}{\pi R} \int_{x_1}^{x_i} c \left\{ (x + \mu \sin \Psi)^2 + \frac{1}{2} \left(v_a - \frac{v_{i(x,\Psi)}}{\Omega R}\right)^2 \right\} dx \tag{110}$$

Using the first two terms of each series of equation (11) and equation (12) and noting that values of c_{k_o} and s_{k_o} other than zero would result in a jump in axial velocity across the rotor hub, which is physically impossible, gives:

$$\frac{v_{i(x,\Psi)}}{\Omega R} = c_{oo} + c_{o1} x + c_{12} x^2 + c_{K1} x \cos \Psi$$
$$+ s_{K1} x \sin \Psi \tag{111}$$

Now substitution of equation (97), which contains the principle effects of circulation, (5) and (111) into equations (104), (110), and (107); and then integrating with respect to x utilizing equations (111) and (37) gives equations (112), (113), and (114). These last three equations are written in tabular form where the coefficients in the boxes must be multiplied by row and column heads.

$$\frac{(\Delta c_{xy})_a}{a} = \tag{112}$$

	σ_1	σ_2	σ_3	σ_4	σ_5
1	$v_a^* (v_a^* - \frac{1}{2}\mu a_1^*) + \frac{a_o^2 \mu^2}{2}$	$-c_{o1}(v_a^* - \frac{1}{2}\mu a_1^*) + (v_a^* - \frac{s_{K1}}{\mu^2})(A_o^* + \frac{3}{4}\theta_1) + (c_{K1} - b_1^* \frac{a_o \mu}{2}$	$-c_{o2}(v_a^* - \frac{1}{2}\mu a_1^*) - c_{o1}(A_o^* + \frac{3}{4}\theta_1) - v_a^*\theta_1 + \frac{s_{K1}}{2}(a_1^* + \mu\theta_1) - \frac{c_{K1} b_1^*}{2}$	$-c_{o2}(A_o^* + \frac{3}{4}\theta_1) + c_{o1}\theta_1$	$c_{o2}\theta_1$
$\sin\Psi$	$v_a^* \mu (A_o^* + \frac{3}{4}\theta_1) - \frac{b_1^* a_o \mu^2}{4}$	$-s_{K1}(v_a^* - \frac{1}{2}\mu a_1^*) - c_{o1}\mu(A_o^* + \frac{3}{4}\theta_1) - v_a^*(a_1^* + \mu\theta_1) - (c_{K1} b_1^* - s_{K1} a_1^*)\frac{\mu}{4}$	$-(c_{o2}\mu + s_{K1})(A_o^* + \frac{3}{4}\theta_1) + c_{o1}(a_1^* + \mu\theta_1)$	$s_{K1}\theta_1 + c_{o2}(a_1^* + \mu\theta_1)$	
$\cos\Psi$	$-a_o \mu (v_a^* - \frac{1}{2}\mu a_1^*) - v_a^* a_o \mu - \frac{a_1^* a_o \mu^2}{4}$	$-c_{K1}(v_a^* - \frac{1}{2}\mu a_1^*) - a_o\mu(A_o^* + \frac{3}{4}\theta_1) + c_{o1}(\mu a_o) + v_a^* b_1^* - (c_{K1} a_1^* + s_{K1} b_1^*)\frac{\mu}{4}$	$-c_{K1}(A_o^* + \frac{3}{4}\theta_1) + (\theta_1 + c_{o2}) a_o \mu - c_{o1} b_1^*$	$c_{K1}\theta_1 - c_{o2} b_1^*$	

(continued)

	σ_1	σ_2	σ_3	σ_4	σ_5
sin 2Ψ	$- a_o \mu^2 (A_o^* + \frac{3}{4}\theta_1)$ $+ v_a^* \left(\frac{b_1^* \mu}{2}\right)$	$-\frac{c_{K1}\mu}{2}(A_o^* + \frac{3}{4}\theta_1)$ $+\frac{a_o \mu}{2}(a_1^* + \mu\theta_1)$ $-(c_{o1} b_1^* - s_{K1} a_o)\frac{\mu}{2}$	$\frac{c_{K1}}{2}(a_1^* + \mu\theta_1)$ $-(c_{o2}\mu + s_{K1})\frac{b_1^*}{2}$		
cos 2Ψ	$v_a^* \left(\frac{a_1^* \mu}{2}\right)$ $+\frac{a_o^2\mu^2}{2}$	$-\frac{s_{K1}\mu}{2}(A_o^* + \frac{3}{4}\theta_1)$ $(c_{K1} - b_1^*)\frac{a_o\mu}{2}$ $-\frac{c_{o1}a_1^*\mu}{2}$	$-\frac{s_{K1}}{2}(a_1^* + \mu\theta_1)$ $-\frac{c_{o2}a_1^*\mu}{2}$ $-\frac{c_{K1}b_1^*}{2}$		
sin 3Ψ	$-\frac{b_1^* a_o \mu^2}{4}$	$-(c_{K1}b_1^* + s_{K1}a_1^*)\frac{\mu}{4}$			
cos 3Ψ	$-\frac{a_1^* a_o \mu^2}{4}$	$-(c_{K1}a_1^* - s_{K1}b_1^*)\frac{\mu}{4}$			

Equation 112 (concluded)

$$\frac{-(\Delta c_{xy})_{\delta_o}}{\delta_o} = \qquad (113)$$

	σ_1	σ_2	σ_3	σ_4	σ_5
1	$\frac{1}{2}(v_a^{*2} + \mu^2)$	$-v_a^* c_{o1}$	$1 - v_a^* c_{o2} + \frac{c_{o1}^2}{2}$	$c_{o1} c_{o2}$	$\frac{c_{o2}^2}{2}$
$\sin\Psi$		$2\mu - v_a^* s_{K1}$	$c_{o1} s_{K1}$	$c_{o2} s_{K1}$	
$\cos\Psi$		$-v_a^* c_{K1}$	$c_{o1} c_{K1}$	$c_{o2} c_{K1}$	
$\sin 2\Psi$			$\frac{1}{2} c_{K1} s_{K1}$		
$\cos 2\Psi$	$-\frac{\mu^2}{2}$		$\frac{c_{K1}^2}{4} - \frac{s_{K1}^2}{4}$		

$$\frac{-(\Delta c_{xy})_\varepsilon}{\varepsilon a^2} = \tag{114}$$

	σ_1	σ_2	σ_3	σ_4	σ_5
1	$(v_a^* - \frac{1}{2}\mu a_1^*)^2 + \frac{\mu^2}{2}(A_o^* + \frac{3}{4}\theta_1)^2 + \frac{a_o^2\mu^2}{2} + \frac{\mu^2}{8}(a_1^{*2} + b_1^{*2})$	$2(v_a^* - \frac{1}{2}\mu a_1^*)(A_o^* + \frac{3}{4}\theta_1) - \mu(A_o^* + \frac{3}{4}\theta_1)(a_1^* + \mu\theta_1) - b_1^* a_o \mu$	$-2\theta_1(v_a^* - \frac{1}{2}\mu a_1^*) + (A_o^* + \frac{3}{4}\theta_1)^2 + \frac{1}{2}(a_1^* + \mu\theta_1) + \frac{1}{2}(b_1^*)^2$	$-2(A_o^* + \frac{3}{4}\theta_1)\theta_1$	θ_1^2
sin Ψ	$2\mu(v_a^* - \frac{1}{2}\mu a_1^*)(A_o^* + \frac{3}{4}\theta_1) - \frac{1}{2}a_1^*\mu^2(A_o^* + \frac{3}{4}\theta_1) - \frac{1}{2}a_o b_1^*\mu^2$	$-2(v_a^* - \frac{1}{2}\mu a_1^*)(a_1^* + \mu\theta_1) + 2\mu(A_o^* + \frac{3}{4}\theta_1)^2 + \frac{1}{2}a_1^*\mu(a_1^* + \mu\theta_1) + \frac{1}{2}b_1^{*2}\mu$	$-2(a_1^* + 2\mu\theta_1)(A_o^* + \frac{3}{4}\theta_1)$	$2\theta_1(a_1^* + \mu\theta_1)$	

(Continued)

	σ_1	σ_2	σ_3	σ_4	σ_5
cos Ψ	$-2a_o\mu(v_a^* - \frac{1}{2}\mu a_1^*)$ $+ \frac{1}{2} b_1^* \mu^2(A_o^* + \frac{3}{4}\theta_1)$ $- \frac{1}{2} a_o a_1^* \mu^2$	$2b_1^*(v_a^* - \frac{1}{2}\mu a_1^*)$ $-2a_o\mu(A_o^* + \frac{3}{4}\theta_1)$ $- \frac{1}{2} b_1^*\mu(a_1^* + \mu\theta_1)$ $+ \frac{1}{2} a_1^* b_1^* \mu$	$2b_1^*(A_o^* + \frac{3}{4}\theta_1)$ $+ 2a_o\mu\theta_1$	$-2b_1^*\theta_1$	
sin 2Ψ	$b_1^*\mu(v_a^* - \frac{1}{2}\mu a_1^*)$ $-a_o\mu^2(A_o^* + \frac{3}{4}\theta_1)$	$2b_1^*\mu(A_o^* + \frac{3}{4}\theta_1)$ $+ a_o\mu(a_1^* + \mu\theta_1)$	$-b_1^*(a_1^* + 2\mu\theta_1)$		
cos 2Ψ	$a_1^*\mu(v_a^* - \frac{1}{2}\mu a_1^*)$ $- \frac{\mu^2}{2}(A_o^* + \frac{3}{4}\theta_1)^2$ $+ \frac{a_o^2\mu^2}{2}$	$a_1^*\mu(A_o^* + \frac{3}{4}\theta_1)$ $+ \mu(A_o^* + \frac{3}{4}\theta_1)(a_1^*$ $+ \mu\theta_1) - a_o b_1^*\mu$	$-\frac{1}{2}(a_1^* + \mu\theta_1)$ $- a_1^*\mu\theta_1$ $+ \frac{b_1^{*2}}{2}$		

(Continued)

	σ_1	σ_2	σ_3	σ_4	σ_5
sin 3 Ψ	$\frac{1}{2} a_1^* \mu^2 (A_o^* + \frac{3}{4} \theta_1)$ $- \frac{a_o b_1^* \mu^2}{2}$	$- \frac{1}{2} a_1^* \mu (a_1^* + \mu \theta_1)$ $+ \frac{1}{2} b_1^{*2} \mu$			
cos 3 Ψ	$- \frac{1}{2} b_1^* \mu^2 (A_o^* + \frac{3}{4} \theta_1)$ $- \frac{1}{2} a_o a_1^* \mu^2$	$\frac{1}{2} a_1^* b_1^* \mu$ $+ \frac{1}{2} b_1^* \mu (a_1^* + \mu \theta_1)$			
sin 4 Ψ	$\frac{1}{4} a_1^* b_1^* \mu^2$				
cos 4 Ψ	$\frac{\mu^2}{8} (a_1^{*2} - b_1^{*2})$				

Equation 114 (concluded)

Equation for the Rotor Torque.--A rotor torque coefficient $C_Q = \dfrac{Q}{\rho \pi \Omega^2 R^5}$ positive in direction opposite to blade rotation, and numerically equal to rotor power coefficient $C_Q = \dfrac{P}{\rho \pi \Omega^3 R^5}$ where Q is average rotor shaft torque, and $P = \Omega Q$ is the rotor shaft power is defined.

Then

$$\frac{2C_Q}{b} = -\frac{1}{2\pi}\int_0^{2\pi}\int_{x_1}^{x_i} \Delta C_{xy}\, x\, d\,\Psi\, dx \tag{115}$$

or

$$\frac{2C_Q}{b} = -\,(\text{constant term in } C_{xy} \text{ with } \sigma_n = \sigma_{n+1}) \tag{116}$$

which gives from equations (112), (113), and (114) the following:

$$\begin{aligned}
\frac{2C_Q}{ab\,\sigma_4} = &\Bigg\{\left[-v_a^*(v_a^* - \tfrac{1}{2}\mu a_1^*) - \frac{a_o^2\mu^2}{2}\right]\frac{\sigma_2}{\sigma_4} + \Bigg[c_{o1}(v_a^* - \tfrac{1}{2}a_1^*) \\
&- (v_a^* - \frac{s_{K1}\mu}{2})(A_o^* + \tfrac{3}{4}\theta_1) - (c_{K1} - b_1^*)\frac{a_o\mu}{2}\Bigg]\frac{\sigma_3}{\sigma_4} \\
&+ \Bigg[c_{o2}(v_a^* - \tfrac{1}{2}\mu a_1^*) + c_{o1}(A_o^* + \tfrac{3}{4}\theta_1) + v_a^*\theta_1 - \frac{s_{K1}}{2}(a_1^* + \mu\theta_1) \\
&+ \frac{c_{K1}b_1^*}{2}\Bigg] + \left[c_{o2}(A_o^* + \tfrac{3}{4}\theta_1) - c_{o1}\theta_1\right]\frac{\sigma_5}{\sigma_4} - c_{o2}\theta_1\frac{\sigma_6}{\sigma_4}\Bigg\} \\
&+ \frac{\delta_o}{a}\Bigg\{\frac{1}{2}(v_a^{*2} + \mu^2)\frac{\sigma_2}{\sigma_4} - (v_a^* c_{o1})\frac{\sigma_3}{\sigma_4} + \Bigg[1 - v_a^* c_{o2} + \frac{c_{o1}^2}{2} \\
&+ \frac{s_{K1}^2}{4} + \frac{c_{K1}^2}{4}\Bigg] + c_{o1}c_{o2}\frac{\sigma_5}{\sigma_4} + \frac{c_{o2}^2}{2}\frac{\sigma_6}{\sigma_4}\Bigg\} + \varepsilon a\Bigg\{\Bigg[(v_a^* - \tfrac{1}{2}\mu a_1^*)^2 \\
&+ \frac{\mu^2}{2}(A_o^* + \tfrac{3}{4}\theta_1)^2 + \frac{a_o^2\mu^2}{2} + \frac{\mu^2}{8}(a_1^{*2} + b_1^{*2})\Bigg]\frac{\sigma_2}{\sigma_4} +
\end{aligned} \tag{117}$$

$$+ \left[2(v_a^* - \frac{1}{2}\mu a_1^*)(A_o^* + \frac{3}{4}\theta_1) - \mu(A_o^* + \frac{3}{4}\theta_1)(a_1^* + \mu\theta_1)\right.$$

$$\left. - b_1^* a_o \mu\right]\frac{\sigma_3}{\sigma_4} + \left[-2\theta_1(v_a^* - \frac{1}{2}\mu a_1^*) + (A_o^* + \frac{3}{4}\theta_1)^2\right.$$

$$\left. + \frac{1}{2}(a_1^* + \mu\theta_1) + \frac{1}{2}b_1^{*2}\right] - 2(A_o^* + \frac{3}{4}\theta_1)\theta_1\left(\frac{\sigma_5}{\sigma_4}\right)$$

$$\left. + \theta_1^2\left(\frac{\sigma_6}{\sigma_4}\right)\right\} + \Delta C_{Q_s}$$

To evaluate ΔC_{Q_s} in equation (117) the following is a development parallel to that presented by Stevens (3) and Castles (4).

The tip stall increment of torque coefficient, ΔC_{Q_s}, occurs at large values of μ due to tip stall on the retreating blade. The small negative increment to C_Q of the profile drag on the blade elements within the reverse flow region is neglected. The value of the increment ΔC_{Q_s}, to the rotor torque coefficient can be approximately determined by considering the blade angle of attack, α_R, on the retreating blade at non-dimensional radius x and azimuth angle $\Psi = \frac{3\pi}{2}$.

$$\alpha_R = \theta + \varphi = A_o^* + a_1^* + \Delta\theta_t + \frac{v_a - x(c_{o1} + s_{11})}{x - \mu} \qquad (118)$$

which approximates φ from equation (109) and $\frac{V_{i(x,\Psi)}}{R}$ from equation (111) after taking c_{oo} and $c_{o2} = 0$ as a result of their undetermined values. Taking $c_{oo} = 0$ also allows v_a^* to be taken equal to v_a. $\Delta\theta_t$ = blade twist from x = 0.75 to x = 1, positive for increased blade angle at the blade tip.

Let c_{ℓ_m} denote the blade maximum lift coefficient which normally is of the order of $c_{\ell_m} = 1.2$. Then the local angle of attack at which a given blade element stalls is

$$\alpha_s = \frac{c_{\ell m}}{2\pi} = A_o^* + a_1^* + \Delta\theta_t + \frac{v_a^* - x_s(c_{o1} + s_{K1})}{x_s - \mu} \tag{119}$$

It follows from equation (119) that the non-dimensional radius, x_s, outboard of which the blade is stalled is

$$x_s = \frac{v_a^* + \mu\left(\frac{c_{\ell m}}{2\pi} - A_o^* - a_1^* - \Delta\theta_t\right)}{c_{o1} + s_{K1} + \left(\frac{c_{\ell m}}{2\pi} - A_o^* - a_1^* - \Delta\theta_t\right)} \tag{120}$$

The onset of tip stall on the retreating blade will occur when $x_s = 1$. Thus setting $x_s = 1$ in equation (120) and solving for μ gives

$$(\mu)_s = \frac{c_{o1} + s_{K1} - v_a^* + \left(\frac{c_{\ell m}}{2\pi} - A_o^* - a_1^* - \Delta\theta_t\right)}{\left(\frac{c_{\ell m}}{2\pi} - A_o^* - a_1^* - \Delta\theta_t\right)} \tag{121}$$

The portion of the rotor disk within which the tip blade elements of the retreating blade will be stalled is very nearly a segment of minimum radius x_s. The area of this segment may be approximated by the inscribed triangles since x_s is never very much smaller than unity.

$$\text{Area of stall} = (1 - x_s)\sqrt{1 - x_s^2} \tag{122}$$

The incremental torque of a stalled blade-element at $\Psi = \frac{3\pi}{2}$; $x_s = 1$ is $\Delta Q = \frac{1}{2}\rho\Omega^2R^3(1-\mu)^2 \, \Delta c_{d_o} \, c_t \, dr$ (123)

where Δc_{d_o} = average increase in profile drag coefficient for several degrees beyond the angle of attack for stall (the order of $\Delta c_{d_o} = 0.08$).

Consider that on the average the blade area within the stall region is approximately equal to the total area of the blades outboard of x_s, i.e.

$$\text{Area of blades outboard of } x_s = bcR(1 - x_s) \tag{124}$$

times the ratio of the area of the stall segment (given by equation (107)) to the rotor area in the annulus outboard of x_s given by

$$\text{Area in annulus} = \pi(1 - x_s^2)\, R^2 \tag{125}$$

Thus

$$\Delta Q_{Total} = \frac{1}{2}\, \rho\, \Omega^2 R^4\, (1 - \mu)^2\, \Delta c_{d_o}\, b\, c_t\, (1 - x_s)\sqrt{\frac{1-x_s}{1+x_s}} \tag{126}$$

or

$$\Delta C_{Q_s} = \frac{\Delta cd_o}{2\,\pi}\, \sigma_t\, (1 - \mu)^2 (1 - x_s)\, \sqrt{\frac{1 - x_s}{1 + x_s}} \tag{127}$$

where σ_t = tip solidity = $\frac{bc_t}{\pi R}$.

If x_s is less than μ the stall point is within the reverse flow region, and if x_s is greater than μ no outer blade-elements are stalled. It is convenient to develop an expression for the section lift coefficient at the tip to determine if there is a stall region, especially when considering design criteria.

$$c_{\ell T} = a\,\alpha_{tip} \tag{128}$$

and

$$\alpha_{tip} = \theta + \varphi \quad \text{evaluated at} \quad \Psi = \frac{3\pi}{2}; \quad x = 1 \text{ gives:}$$

$$\alpha_{tip} = A_o^* + a_1^* + \tan^{-1} \frac{v_a^*}{1-\mu} \tag{129}$$

Considering $\frac{V_{i(x,\ \Psi)}}{\Omega R}$ as having a small effect at the μ's where tip stall is usually encountered, substitution into equation (128) from equations (90) and (91) gives:

$$\frac{c_{\ell_t}}{a} = \frac{(1+\frac{3}{2}\mu^2)\bar{\alpha} - \frac{3}{8}\mu^2(1+\frac{3}{2}\mu^2)\theta_1 - \frac{3}{2}(1-\frac{1}{2}\mu^2)v_a^* + \mu\left[\frac{8}{3}\bar{\alpha} - \mu^2\theta_1 - 2(1-\frac{3}{2}\mu^2)v_a^*\right]}{(1+\frac{3}{2}\mu^2)^2 - 4\mu^2}$$

$$+ \tan^{-1}\left[\frac{v_a^*}{1-\mu}\right] \tag{130}$$

where v_a^* is usually taken $= v_a$.

<u>Equation for the Rotor X-Force.</u>--An X-force coefficient

$$C_x = \frac{F_x}{\frac{1}{2}\rho\ \pi\Omega^2 R^4} \tag{131}$$

is defined, then:

$$C_x = -\frac{b}{2\pi}\int_0^2\int_{x_i}^{x_1}\Delta C_{xy}\ \sin\ \Psi\ d\Psi\ dx \tag{132}$$

or

$$\frac{2C_x}{b} = -\text{(coefficient of the sin } \Psi \text{ terms in } C_{xy} \text{ in equations (112), (113) and (114))} \tag{133}$$

Thus

$$\frac{2C_x}{ab\sigma_3} = \left\{-\left[v_a^*\ \mu(A_o^* + \frac{3}{4}\theta_1) - \frac{b_1^* a_o\ \mu^2}{4}\right]\frac{\sigma_1}{\sigma_3} + \left[s_{K1}(v_a^*\right.\right. \tag{134}$$

$$- \frac{1}{2}\mu a_1^*) + c_{o1}\ \mu(A_o^* + \frac{3}{4}\theta_1) + v_a^*(a_1^* + \mu\theta_1) +$$

$$+ (c_{K1}b_1^* - s_{K1}a_1^*)\frac{\mu}{4}\Bigg]\frac{\sigma_2}{\sigma_3} + \Bigg[(c_{o2}\mu + s_{K1})(A_o^* + \frac{3}{4}\theta_1)$$

$$- c_{o1}(a_1^* + \mu\theta_1)\Bigg] - \Bigg[s_{K1}\theta_1 + c_{o2}(a_1^* + \mu\theta_1)\Bigg]\frac{\sigma_4}{\sigma_3}\Bigg\}$$

$$+ \frac{\delta_o}{a}\Bigg\{\Bigg[2\mu - v_a^* s_{K1}\Bigg]\frac{\sigma_2}{\sigma_3} + \Bigg[c_{o1}s_{K1}\Bigg] + \Bigg[c_{o2}s_{K1}\Bigg]\frac{\sigma_4}{\sigma_3}\Bigg\}$$

$$+ \varepsilon a\Bigg\{\Bigg[2\mu(v_a^* - \frac{1}{2}\mu a_1^*)(A_o^* + \frac{3}{4}\theta_1) - \frac{1}{2}a_1^*\mu^2(A_o^* + \frac{3}{4}\theta_1)$$

$$- \frac{1}{2}a_o b_1^*\mu^2\Bigg]\frac{\sigma_1}{\sigma_3} + \Bigg[-2(v_a^* - \frac{1}{2}\mu a_1^*)(a_1^* + \mu\theta_1) + 2\mu(A_o^* + \frac{3}{4}\theta_1)^2$$

$$+ \frac{1}{2}a_1^*\mu(a_1^* + \mu\theta_1) + \frac{1}{2}(b_1^*)^2\mu\Bigg]\frac{\sigma_2}{\sigma_3} - 2(a_1^* + 2\mu\theta_1)(A_o^*$$

$$+ \frac{3}{4}\theta_1) + 2\theta_1(a_1^* + \mu\theta_1)\quad\frac{\sigma_4}{\sigma_3}\Bigg\}$$

Equation for the Rotor Y-Force.--A rotor Y-force coefficient

$$C_y = \frac{F_y}{\frac{1}{2}\rho\pi\Omega^2R^4} \tag{135}$$

is defined, then:

$$C_y = \frac{b}{2\pi}\int_o^{2\pi}\int_{x_i}^{x_1}\Delta C_{xy}\cos\Psi\, d\Psi\, dx \tag{136}$$

or

$$\frac{2C_y}{b} = (\text{coefficient of }\cos\Psi\text{ terms in }C_{xy}\text{ in equations (112), (113) and (114)}) \tag{137}$$

As the dynamic pressure distribution over the rotor radii is essentially the same for blades in opposite positions, 180° apart, in front and back of rotor, the contribution of profile drag forces is small, hence neglected. Thus:

$$\frac{2C_y}{ab\sigma_3} = -\left\{\left[a_o\ \mu(v_a^* - \frac{1}{2}\mu a_1^*) + v_a^*\ a_o\mu + \frac{a_1^* a_o\ \mu^2}{4}\right]\frac{\sigma_1}{\sigma_3}\right. \tag{138}$$

$$+ \left[c_{K1}(v_a^* - \frac{1}{2}\mu a_1^*) + a_o\mu\ (A_o^* + \frac{3}{4}\theta_1) - c_{o1}\mu\ a_o\right.$$

$$\left. - v_a^* b_1^* + (c_{K1}a_1^* + s_{K1}b_1^*)\ \frac{\mu}{4}\right]\frac{\sigma_2}{\sigma_3} + \left[c_{K1}(A_o^* + \frac{3}{4}\theta_1)\right.$$

$$\left.\left. - (\theta_1 + c_{o2})a_o\mu + c_{o1}b_1^*\right] + \left[c_{o2}b_1^* - c_{K1}\theta_1\right]\frac{\sigma_4}{\sigma_3}\right\}$$

<u>Determination of the Longitudinal Tilt of the Tip-path Plane, θ_y, and the Rotor Angle of Attack α_R.</u>--Castles (2) has shown that these problems can be handled by considering the forces acting on the rotor hub for the general case of steady inclined flight where the flight path is at an angle, φ_c, to the horizontal (positive for descent) and the tip-path plane is at an angle, θ_y, to the horizontal (positive for rearward inclination). See Figure 3.

In helicopter calculations the fuselage lift can usually be neglected. Also the lateral tilt of the tip-path plane has a negligible effect. It follows from the geometry of the forces shown in Figure 4, that

$$\tan\theta_y = \frac{D_F\cos\varphi_c + F_x\cos\theta_y}{W - D_F\sin\varphi_c + F_x\sin\theta_y} \tag{139}$$

where D_F = fuselage drag

W = gross weight of the helicopter

Since the X-force is small compared to the parasite drag for power-on flight conditions, a sufficiently exact solution of equation (139) may be obtained on the reiteration using as a first approximation

$$\tan \theta_y \approx \frac{D_F \cos \varphi_c}{W - D_F \sin \varphi_c} \tag{140}$$

Knowing the flight path inclination and the longitudinal tilt of the tip-path plane, the rotor angle of attack can be determined.

$$\alpha_r = \varphi_c + \theta_y \tag{141}$$

Determination of the Rotor Thrust Coefficient.--Since the longitudinal tilt of the tip-path plane is always a small angle for steady state flight, an equation for the rotor thrust may be determined by equating to zero the summation of forces acting on the rotor in the vertical direction. Again referring to Figure 3 and neglecting fuselage lift, an expression for the thrust force, T, is

$$T = \frac{W - D_F \sin \varphi_c + F_x \sin \theta_y}{\cos \theta_y} \tag{142}$$

Substitution of the above equation into the equation for the rotor thrust coefficient given by equation (31) gives

$$C_T = \frac{W - D_F \sin \varphi_c + F_x \sin \theta_y}{\rho \pi \Omega^2 R^4 \cos \theta_y} \tag{143}$$

CHAPTER III

APPLICATION

Performance Charts and Equations.--Although there is nothing to preclude the use of the equations in Chapter II directly for the computation of performance data, a method is presented in this Chapter which greatly reduces the labor involved for those who do not have the use of a computer

The values of the coefficients of the representation of the induced velocity given in equation (111) are taken to be as follows as were recommended by Castles (6):

$$c_{oo} = c_{o2} = 0 \tag{144}$$

since their value is undetermined in the present "state of the art," and

$$c_{o1} \approx \frac{\frac{3}{4} C_T}{(1 - \mu^2)\sqrt{(v_a - \frac{3}{4} c_{o1})^2 + \mu^2}} \tag{145}$$

$$s_{K1} \approx -2c_{o1}\mu \tag{146}$$

$$c_{K1} \approx c_{o1}\left\{(1 - 1.8\mu^2)\sqrt{1 + \left(\frac{v_a - \frac{3}{4} c_{o1}}{\mu}\right)^2} - \sqrt{\frac{v_a - \frac{3}{4} c_{o1}}{\mu}}^{\,2}\right\} \tag{147}$$

The rotor X-force coefficient from equation (134) is represented by its components and further simplified by equations (36), (37) and (144) as follows

$$C_x = \Delta C_{x_{f1}} + \Delta C_{x_{f2}} + \Delta C_{x_{f3}} + \Delta C_{x_{i1}} + \Delta C_{x_{i2}} \qquad (148)$$

where

$$\Delta C_{x_{f1}} = ab\,\sigma_3 \left[-\frac{3}{2} v_a \mu (A_o^* + \frac{3}{4}\theta_1) + \frac{3}{4} v_a (a_1^* + \mu\theta_1)\right] \qquad (149)$$

$$\Delta C_{x_{f2}} = \delta_o b\,\sigma_3 \left[\frac{3}{2}\mu\right] \qquad (150)$$

$$\Delta C_{x_{f3}} = \varepsilon a^2 b\,\sigma_3 \left\{\frac{3}{2}\left[2\mu(v_a - \frac{1}{2}\mu a_1^*)(A_o^* + \frac{3}{4}\theta_1) - \frac{1}{2} a_1^* \mu^2 (A_o^* + \frac{3}{4}\theta_1)\right] + \frac{3}{4}\left[-2(v_a - \frac{1}{2}\mu a_1^*)(a_1^* + \mu\theta) + 2\mu(A_o^* + \frac{3}{4}\theta_1)^2 + \frac{1}{2} a_1^* \mu (a_1^* + \mu\theta_1)\right] - (a_1^* + 2\mu\theta_1)(A_o^* + \frac{3}{4}\theta_1) + \frac{3}{4}\theta_1(a_1^* + \mu\theta_1)\right\} \qquad (151)$$

$$\Delta C_{x_{i1}} = ab\,\sigma_3 \left\{\frac{3}{4}\left[s_{K1}(v_a - \frac{1}{2}\mu a_1^*) + c_{o1}\mu(A_o^* + \frac{3}{4}\theta_1) - s_{K1} a_1^* + \frac{\mu}{4}\right] + \frac{1}{2}\left[s_{K1}(A_o^* + \frac{3}{4}\theta_1) - c_{o1}(a_1^* + \mu\theta_1)\right] - \frac{3}{8} s_{K1}\theta_1\right\} \qquad (152)$$

$$\Delta C_{x_{i2}} = \delta_o b\,\sigma_3 \left\{-\frac{3}{4} v_a s_{K1} + \frac{c_{o1} s_{K1}}{2}\right\} \qquad (153)$$

In a like manner, the rotor torque coefficient from equation (117) is separated into its fundamental, adjusted induced velocity, and coning angle components; and further simplified by equation (36), (37), and (144) as follows:

$$C_Q = \Delta C_{Q_{f1}} + \Delta C_{Q_{f2}} + \Delta C_{Q_{f3}} + \Delta C_{Q_{i1}} + \Delta C_{Q_{i2}} + \Delta C_{Q_{fa_o}} + \Delta C_{Q_s} \tag{154}$$

where

$$\Delta C_{Q_{f1}} = ab\sigma_4 \left\{- v_a(v_a - \frac{1}{2}\mu a_1^*) + \frac{2}{3}\left[-v_a(A_o^* + \frac{3}{4}\theta_1\right] + \frac{1}{2} v_a \theta_1\right\} \tag{155}$$

$$\Delta C_{Q_{f2}} = \varepsilon a^2 b\sigma_4 \left\{\left[(v_a - \frac{1}{2}\mu a_1^*)^2 + \frac{\mu^2}{2}(A_o^* + \frac{3}{4}\theta_1)^2 + \frac{\mu^2}{8} a_1^{*2}\right] + \frac{2}{3}\left[2(v_a - \frac{1}{2}\mu a_1^*)(A_o^* + \frac{3}{4}\theta_1) - \mu(A_o^* + \frac{3}{4}\theta_1)(a_1^* + \mu\theta_1)\right] + \frac{1}{2}\left[-2\theta_1(v_a - \frac{1}{2}\mu a_1^*) + (A_o^* + \frac{3}{4}\theta_1)^2 + \frac{1}{2}(a_1^* + \mu\theta_1)\right] + \frac{2}{5}\left[-2(A_o^* + \frac{3}{4}\theta_1)\theta_1\right] + \frac{1}{3}\theta_1^2\right\} \tag{156}$$

$$\Delta C_{Q_{f3}} = \frac{1}{2}(v_a^2 + \mu^2)\,\delta_{ob}\sigma_4 \tag{157}$$

$$\Delta C_{Q_{i1}} = ab\sigma_4 \left\{\frac{2}{3}\left[c_{o1}(v_a - \frac{1}{2}a_1^*) + \frac{s_{K1}\mu}{2}(A_o^* + \frac{3}{4}\theta_1)\right] + \frac{1}{2}\left[c_{o1}(A_o^* + \frac{3}{4}\theta_1) - \frac{s_{K1}}{2}(a_1^* + \mu\theta_1)\right] - \frac{2}{5}c_{o1}\theta_1\right\} \tag{158}$$

$$\Delta C_{Q_{i2}} = \delta_o b\sigma_4 \left\{-\frac{2}{3}v_a c_{o1} + \frac{1}{2} + \frac{c_{o1}^2}{4} + \frac{s_{K1}^2}{8} + \frac{c_{K1}^2}{8}\right\} \tag{159}$$

$$\Delta C_{Q_{fa_o}} = a_o^2 \mu^2 ab \sigma_4 (1 - \varepsilon a) \left[\frac{4}{9(1 + \frac{1}{2} \mu^2)} - \frac{1}{2} \right] \qquad (160)$$

$$\Delta C_{Q_s} = \frac{\Delta c_{d_o}}{2\pi} \sigma_t (1 - \mu)^2 (1 - x_s) \sqrt{\frac{1 - x_s}{1 + x_s}} \qquad (127)$$

The solutions to equations represented in graph form in the Appendix are as follows:

Equation (90)	A_o^*	Figure 4
Equation (91)	a_1^*	Figure 5
Equation (149)	$\Delta C_{x_{f1}}$	Figure 6
Equation (151)	$\Delta C_{x_{f3}}$	Figure 7
Equation (155)	$\Delta C_{Q_{f1}}$	Figure 8
Equation (156)	$\Delta C_{Q_{f2}}$	Figure 9
Equation (96)	a_o	Figure 10

The charts cover ranges of μ from 0 through 0.4, $\bar{\alpha}$ from 0.04 through 0.12, v_a from -0.10 through 0.02, for values of $\theta = 0$.

Thus knowing C_T, δ_o, σ_n, μ, and v_a, C_x can be determined from equations and charts.

The value of α_R for determining v_a and the value of C_T must be obtained by iteration as indicated in the development of equations (139) through (141).

The value of the torque coefficient C_Q is then obtained by solving its related equations and use of related charts as indicated in the sample problem.

Power Required.--Knowing the value of the rotor torque for a particular flight condition, the power required for the main rotor is found from:

$$\text{Rotor HP required} = \frac{\Omega Q}{550} \tag{161}$$

Also the power for the counter-torque rotor along with power losses and the power for accessories must be included if the total helicopter engine power is desired. This total power is usually computed by the following:

$$\text{Engine HP required} = \frac{\Omega Q + (\Omega Q)_{c.t.}}{550\,\eta} \tag{162}$$

where η = efficiency of power and drive system to account for all other miscellaneous losses.

$(\Omega Q)_{c.t.}$ = product of the counter-torque rotor torque and the counter-torque mean angular velocity.

Thrust for the counter-torque rotor may be calculated by using the torque required for the main rotor. The thrust for the counter-torque rotor, T_{ct} is:

$$T_{c.t.} = \frac{Q}{\ell} \tag{163}$$

where Q = the main rotor torque and ℓ is the perpendicular distance from the main rotor shaft to the counter-torque rotor hub.

Sample Rotor Performance Calculation.--The problem is to calculate the rotor torque coefficient for a single rotor helicopter traveling at 76 feet per second and climbing at a rate of 8.75 feet per second. The following additional data are given:

$W = 2560$ pounds

$\Omega R = 443$ feet per second

$C_T = 0.00546$

$\sigma = 0.06$

$f = 25.4$ square feet parasite area

$I_1 = 160$ slugs - square feet

$\rho = 0.0023$ slugs per cubic foot

$R = 19$ feet

$b = 3$ blades

$\theta_1 = 0$

Values are from experimental data for run number eleven given in (5).

Calculation steps:

(1) $\varphi_c = \tan^{-1} - \dfrac{8.75}{76} = -6.51°$

(2) $D_F = \dfrac{1}{2}\ \rho\, r^2 f = \dfrac{1}{2}\,(.0023)(76)^2(25.4) = 169$

(3) $\tan\theta = -0.0653 \quad \theta = -3.75°$ from equation (140)

(4) $\alpha_r = -6.51° - 3.75° = -10.26°$ from equation (141)

(5) $v_a = \dfrac{V \sin \alpha_R}{\Omega R} = \dfrac{(76)(-.178)}{443} = -0.0312$

(6) $\mu = \dfrac{V \cos \alpha_R}{\Omega R} = 0.169$

(7) $\bar{a} = \dfrac{2C_T}{ab\,\sigma_3} = \dfrac{2(.00546)}{2\ (3)(.006218)} = .093168$

(8) Calculate C_x:

(a) By interpolation from figures (6 - a,b)

$$\frac{\Delta C_{x_{f1}}}{ab\sigma_3} = -.0000669 \qquad \Delta C_{x_{f1}} = -.00000784$$

(b) $\Delta C_{x_{f2}} = \delta_o b \sigma_3 \left(\frac{3}{2}\mu\right) = .00003777$

where δ_o is assumed = 0.008

(c) Assume ε = .008 and $a = 2\pi$. By interpolation from figures (7 - a,b) gives:

$$\frac{\Delta C_{x_{f3}}}{\varepsilon a^2 b \sigma_3} = -.00222 \qquad \Delta C_{x_{f3}} = -.00001306$$

(d) Solution of equations (145) and (146) give

$$c_{o1} = 0.024$$
$$s_{11} = -.008112$$

and interpolation from figures (4 - b,c) and (5 - a,b) give:

$$A_o^* = 0.138$$
$$a_1^* = 0.0504$$

thus equation (152) gives:

$$\Delta C_{x_{i1}} = -.00010869$$

(e) Solution of equation (153) gives:

$$\Delta C_{x_{i2}} = -.00000001$$

(f) $C_x = -.00009182$ from equation (148)

(9) $F_x = -23.4$ pounds

(10) A recheck of θ_y using Equation (139) gives:

$\tan\theta_y = -.056$ $\qquad \theta_y = -3.21°$

(11) $\alpha_v = -6.51° - 3.21° = -9.72°$ from equation (141)

(12) $v_a = \frac{(76)(-.169)}{443} = -0.0289$

$\mu = \frac{(76)(.9857)}{443} = 0.169$

(13) Reiteration of steps (6) through (11) does not appreciably change the above values. Therefore, the values shown in (12) will be used to compute Rotor Torque.

(14) Calculate C_Q:

(a) Interpolation of appropriate charts gives:

$\Delta C_{Q_{f1}} = .0001554$

$\Delta C_{Q_{f2}} = .00007275$

(b) From equations (157) and (158):

$\Delta C_{Q_{f3}} = .00000153$

$\Delta C_{Q_{i1}} = .0000587$

(c) Equation (159) may be approximated by:

$$\Delta C_{Q_{i2}} \quad \frac{1}{2}\, \delta_o b \sigma_4 = \frac{.000104}{2} = .000052$$

(d) A check of equation (127) indicates $\Delta C_{Q_s} = 0$

(e) $\frac{a_o}{C_T}\left[\frac{I_1 b}{\rho\, \pi R^5}\right] = 0.794$ from figure (10) developed by Stevens (2). $a_o = 0.161$ radians

(f) Equation (160) may be simplified by evaluating the last term in brackets for an average value of μ which gives:

$$\Delta C_{Q_{fa_o}} = \frac{a_o^2 \mu^2 \sigma a}{72} \quad (\varepsilon a - 1) \qquad (164)$$

and $\Delta C_{Q_{fa_o}} = -.000003$

(g) From equation (132): $C_Q = .000337$

Comparison of Results with Experimental Data.--The following values for rotor torque coefficients are compared from the following tabulation:

Run	Flight	MPH	Experimental (5)	Present Theory	Blade-element Theory (2)
11	climb	51.8	0.000359	0.000337	0.000322

Using equations (18), (19), (20), (21), and (141) other parameters are compared:

Parameter	Experimental (5)	Present Theory	Blade-element Theory (2)
A_o	10.00°	9.8°	9.5°
a_1	4.23°	3.6°	3.81°
b_1	3.56°	3.0°	2.88°
a_o	9.15°	9.22°	8.42°
α_v	-9.97°	-9.72°	-9.55°

The following comparison is made for three different flight conditions:

Run	Flight	MPH	Experimental (5)	Present Theory	Blade-element Theory (2)
4	level	58.6	0.000244	0.000261	0.000218
11	climb	51.8	0.000359	0.000337	0.000322
15	auto-rotation 1260 ft. min.	37.7	-0.000008	-0.000006	-0.000016

The rotor blades used in the flight tests were quite flexible permitting dynamic twist which was not considered. In addition the results obtained are greatly dependent on values for induced velocity coefficients at these relatively low values of μ . Another consideration in the comparison, is the estimation of profile drag. The approximation used in the example is probably more appropriate for modern blade design than it is for the blades used in the experiment.

CHAPTER IV

CONCLUSIONS

1. The equations developed in this study, which are based on a fundamental blade bound rotor distribution appear to give good agreement with experimental data.

2. The use of charts and the simplified equations given afford a method for determination of the rotor torque coefficient for a given flight condition in a relatively rapid manner.

3. The charts presented should not need revision with a more accurate representation of the induced velocity coefficients. In addition the appearance of the σ_n factors in the equations will account for any blade chord distribution. It, therefore, appears that this method of estimating helicopter performance should furnish a basis for a more complete treatment whenever additional information is available for the induced velocity coefficients.

APPENDIX

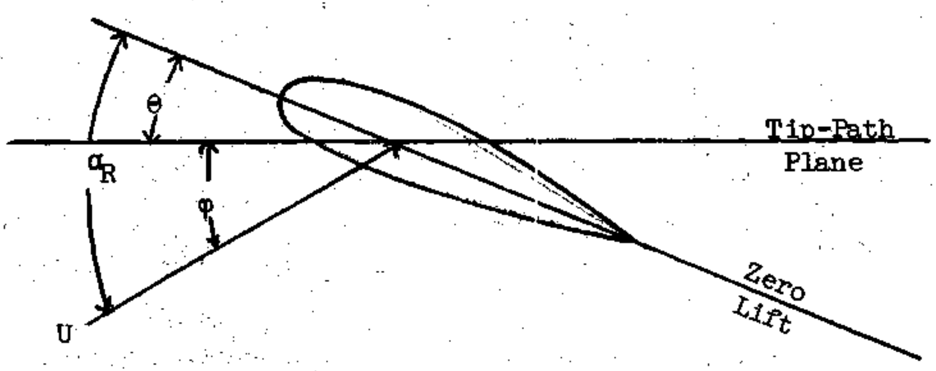


Figure 1. Geometry of Blade Element.

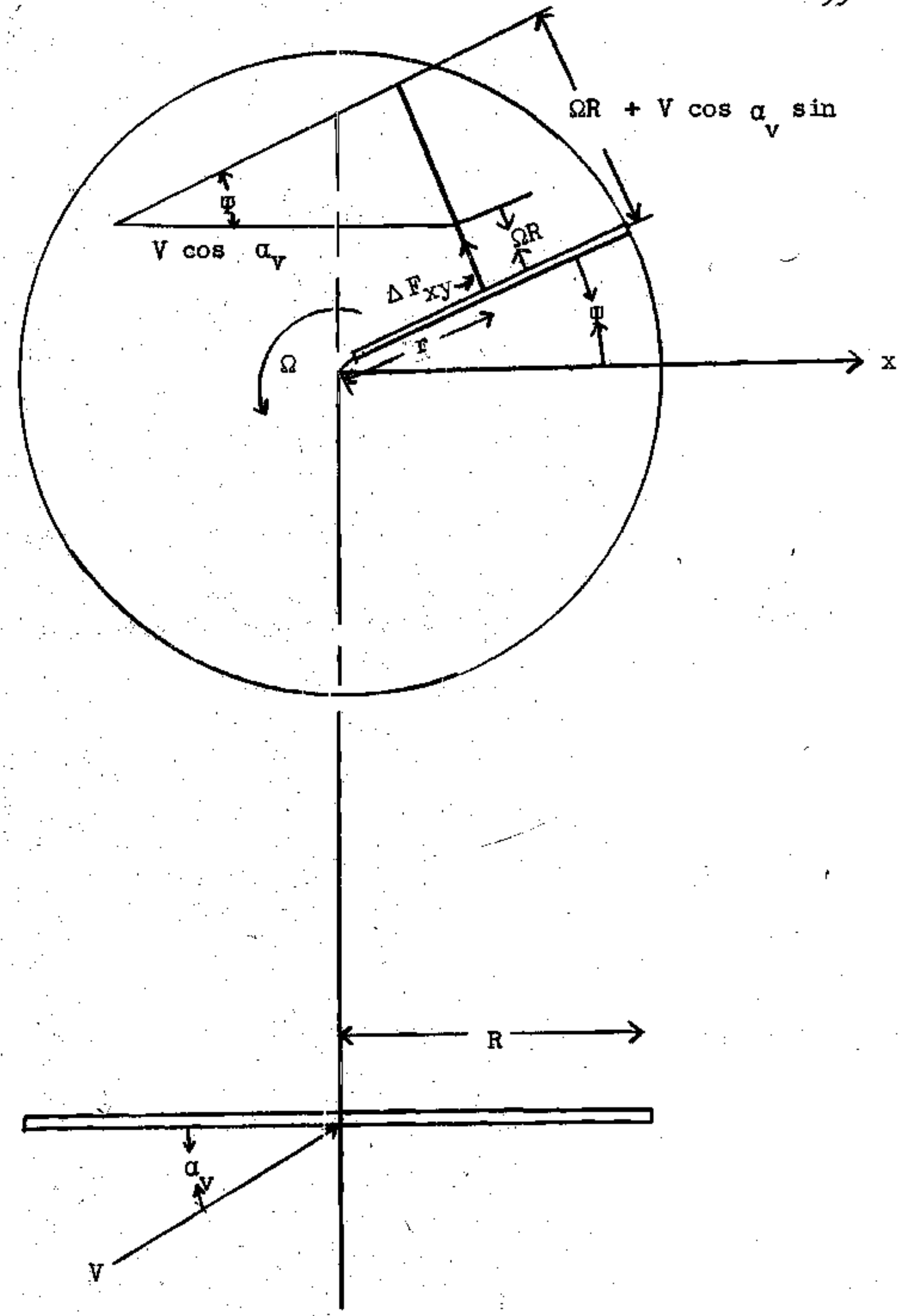
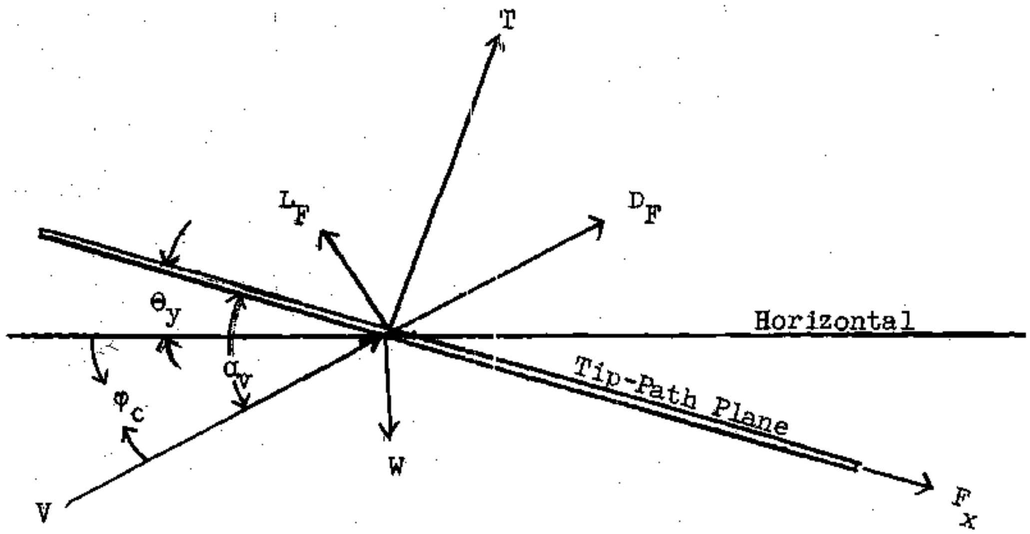


Figure 2. Velocity Components at Blade Element.



All angles and forces shown positive.

Figure 3. Forces on Rotor Hub.

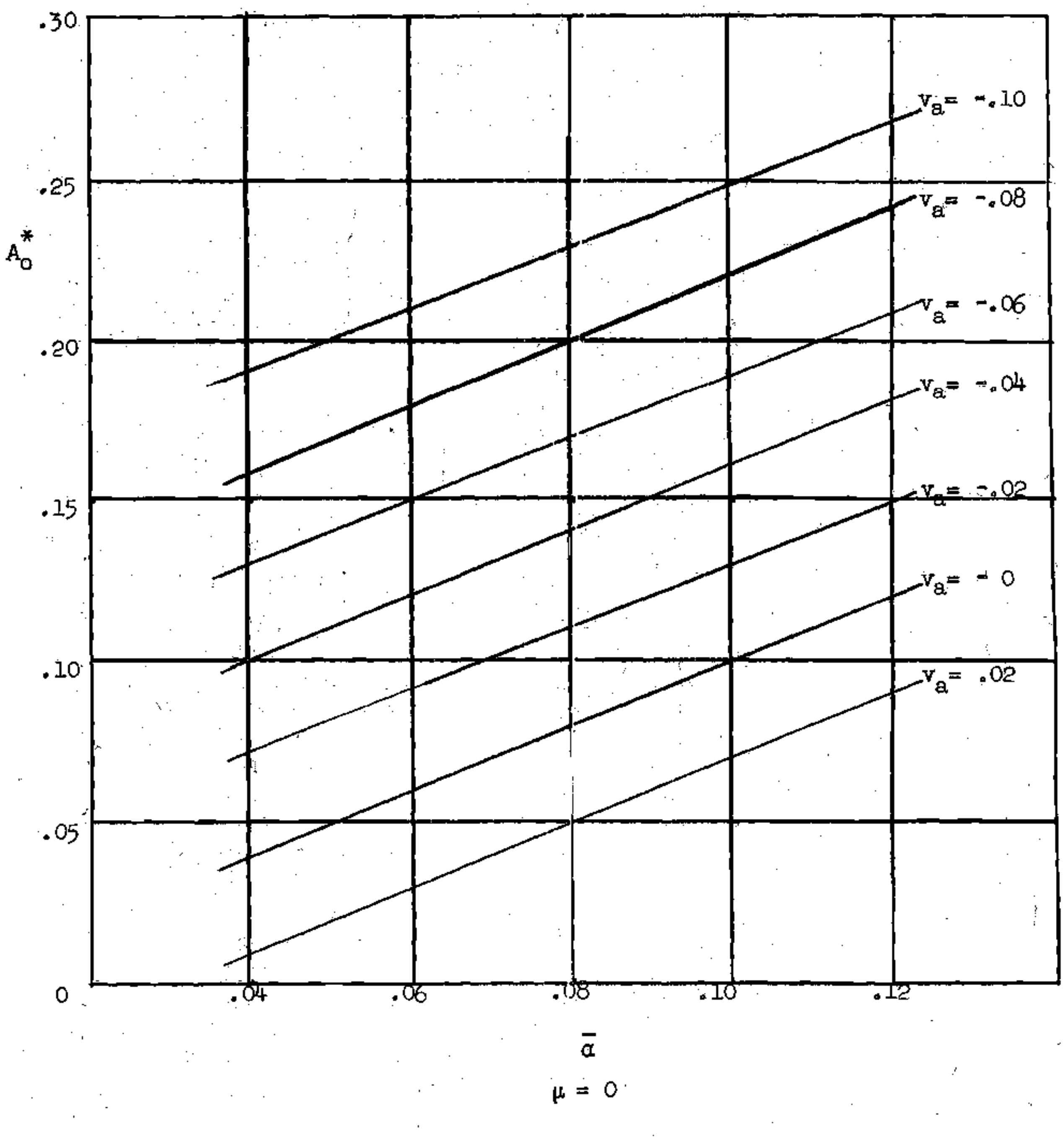


Figure 4(a). Reduced Collective Pitch.

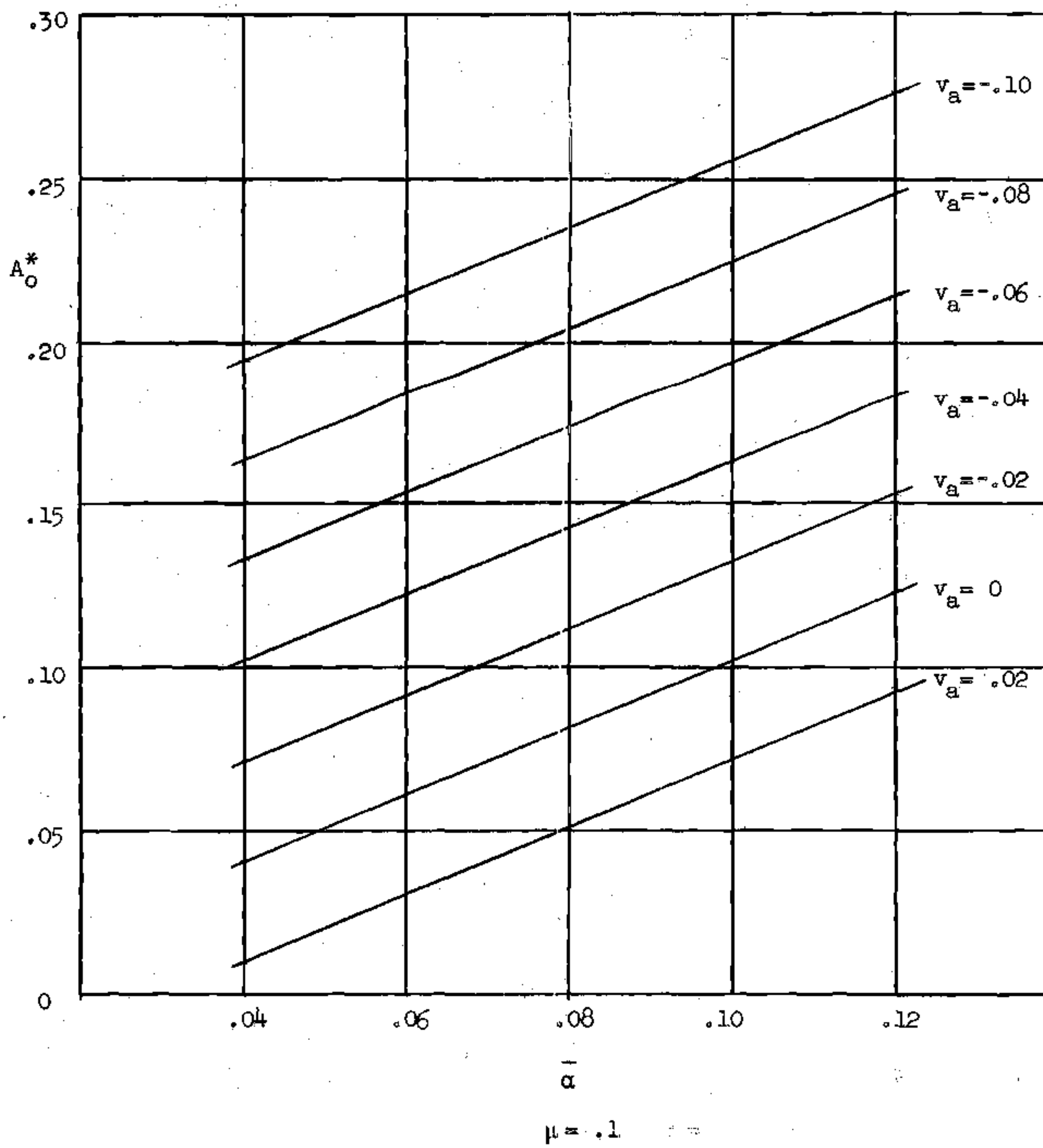


Figure 4(b). Reduced Collective Pitch.

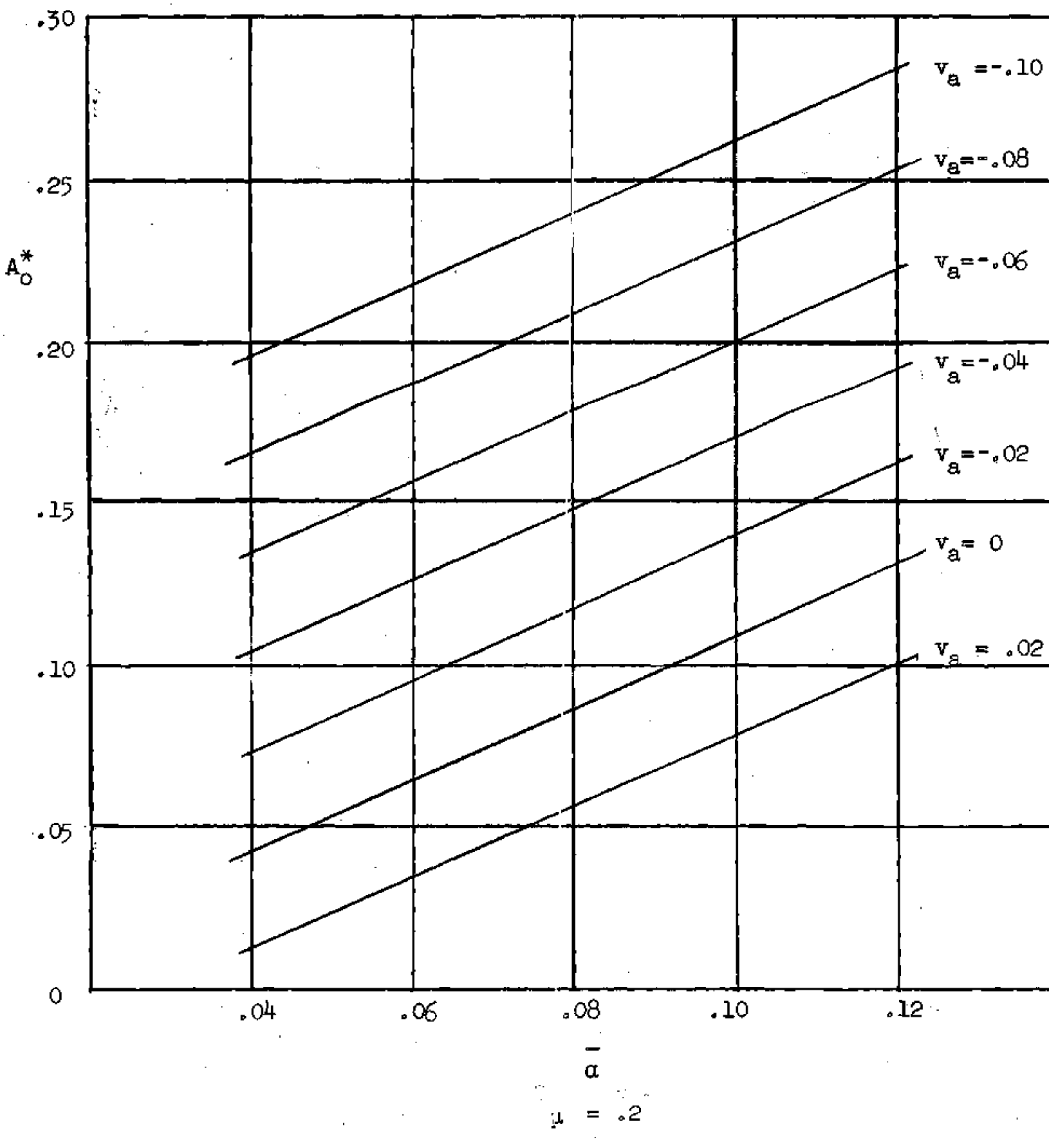
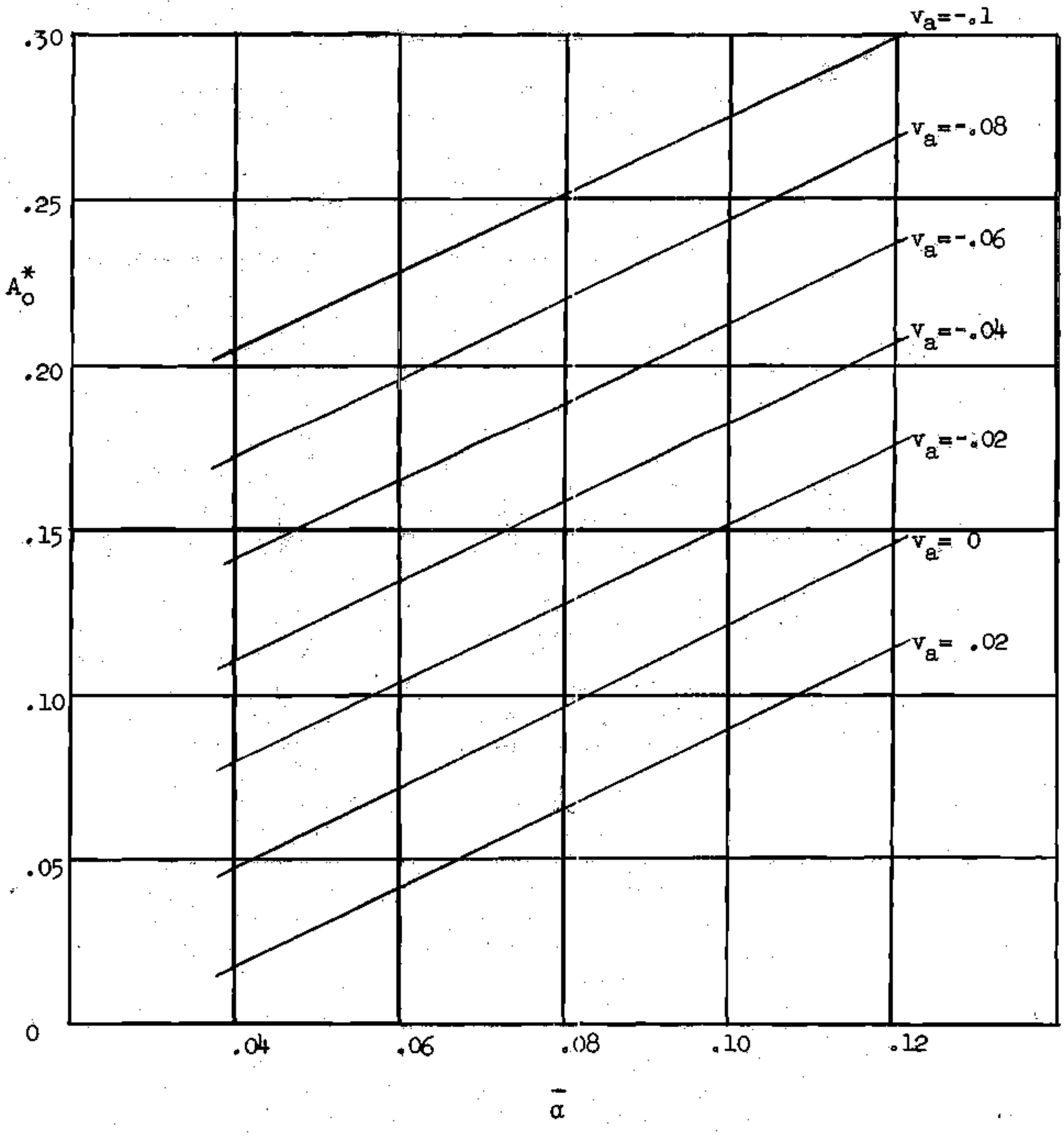


Figure 4(c). Reduced Collective Pitch.



$\mu = .3$

Figure 4(d). Reduced Collective Pitch.

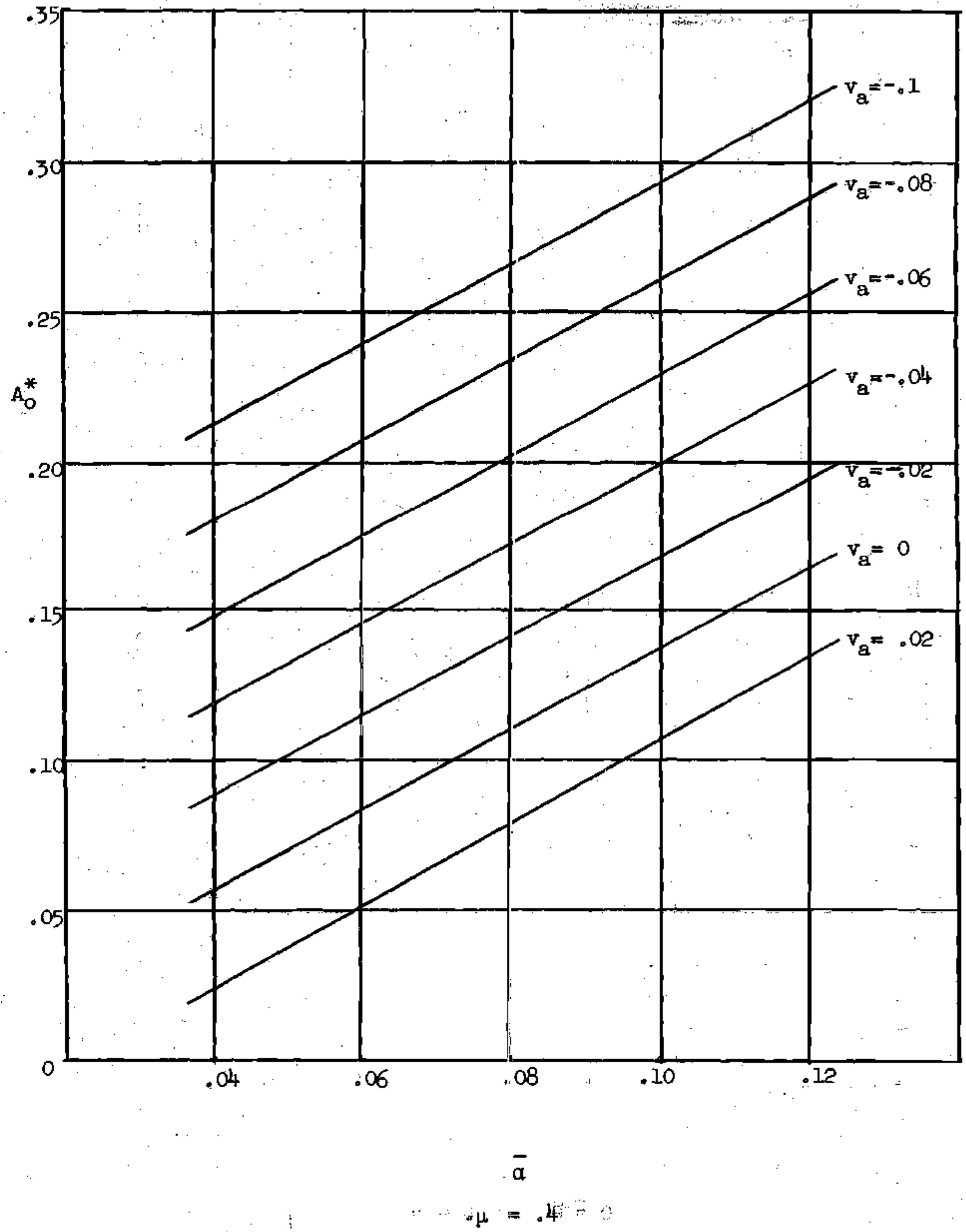


Figure 4(e). Reduced Collective Pitch.

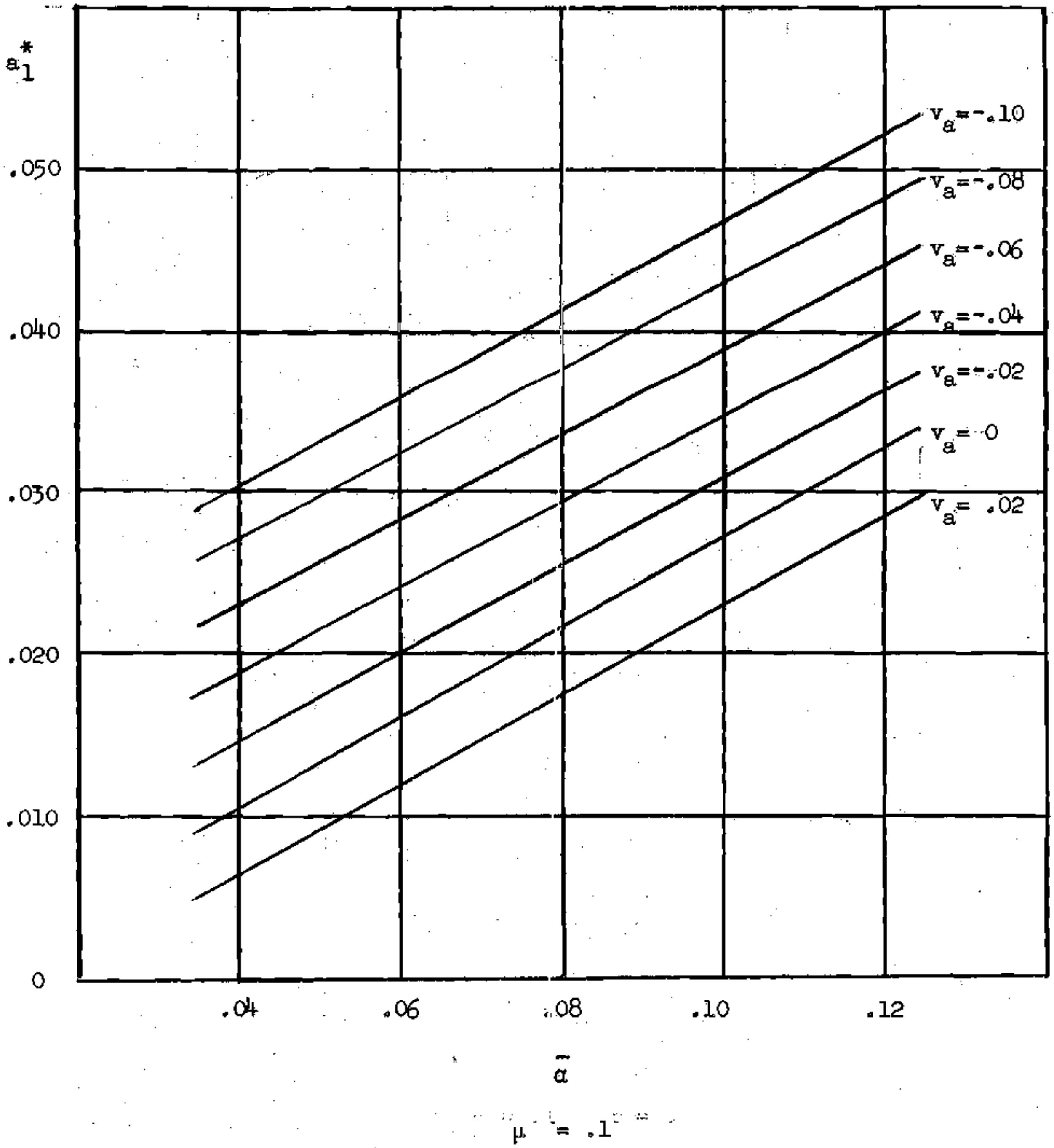


Figure 5(a). Reduced Lateral Cyclic Pitch.

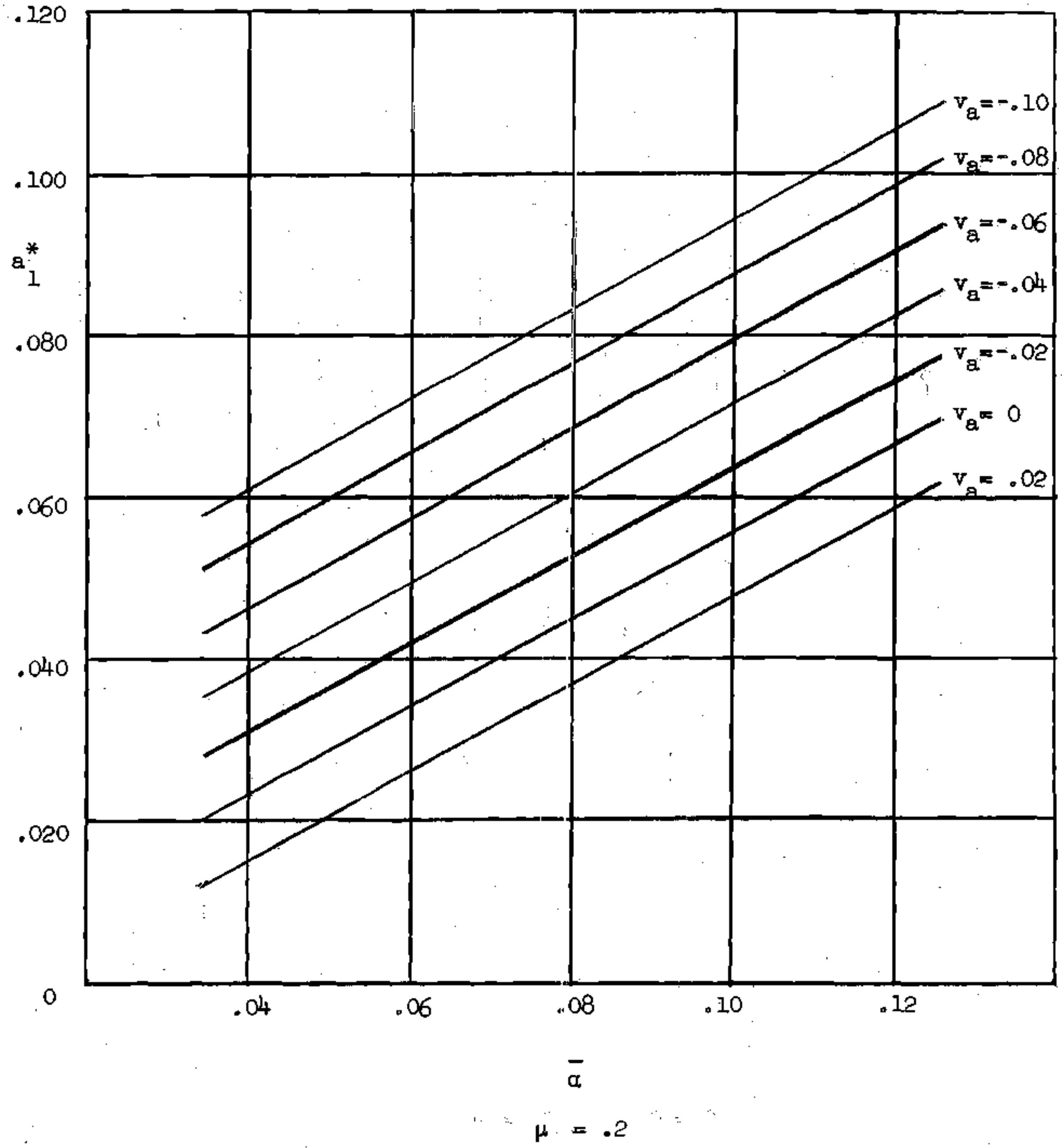
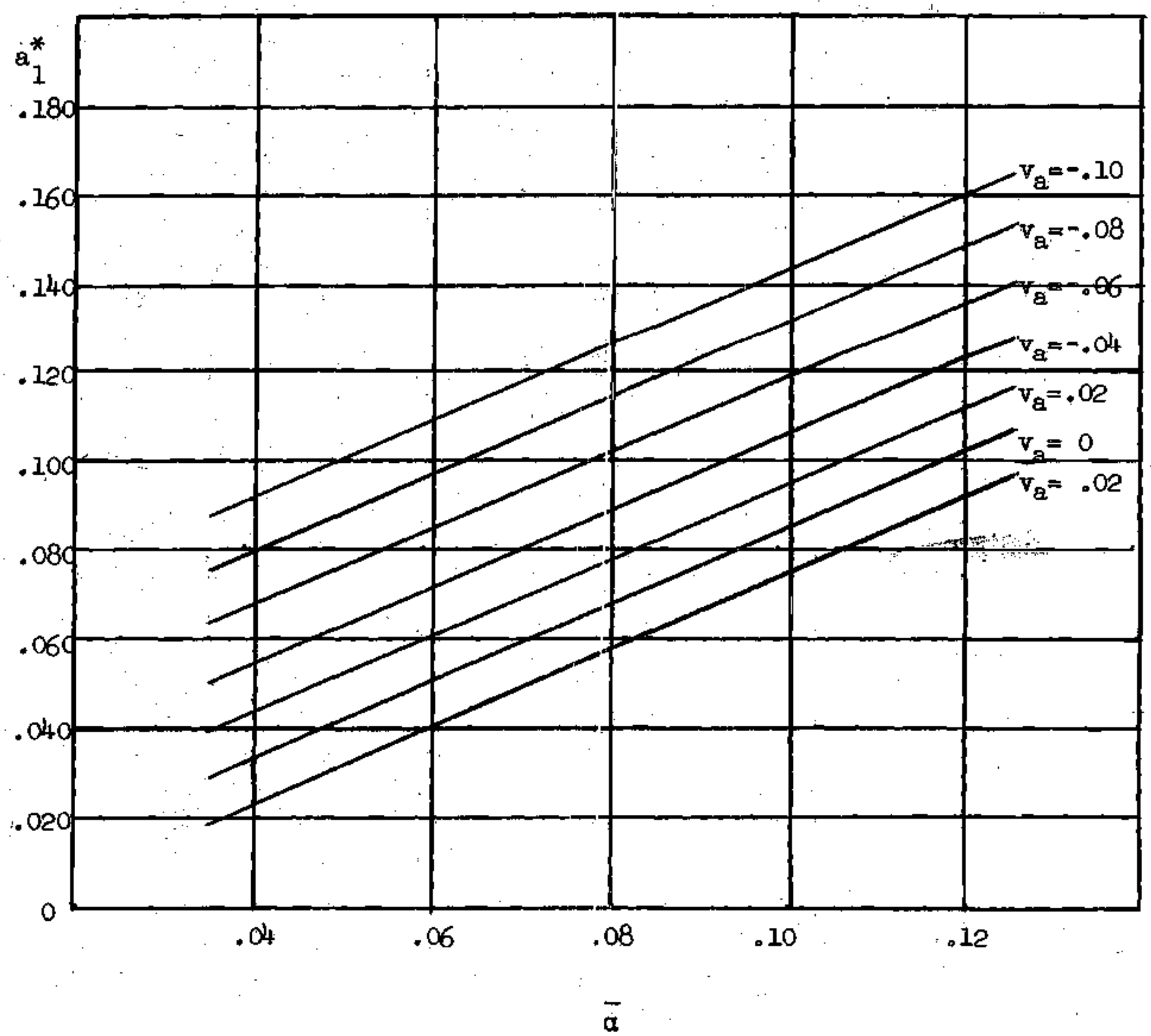


Figure 5(b). Reducei Lateral Cyclic Pitch.



$\mu = .3$

Figure 5(c). Reduced Lateral Cyclic Pitch.

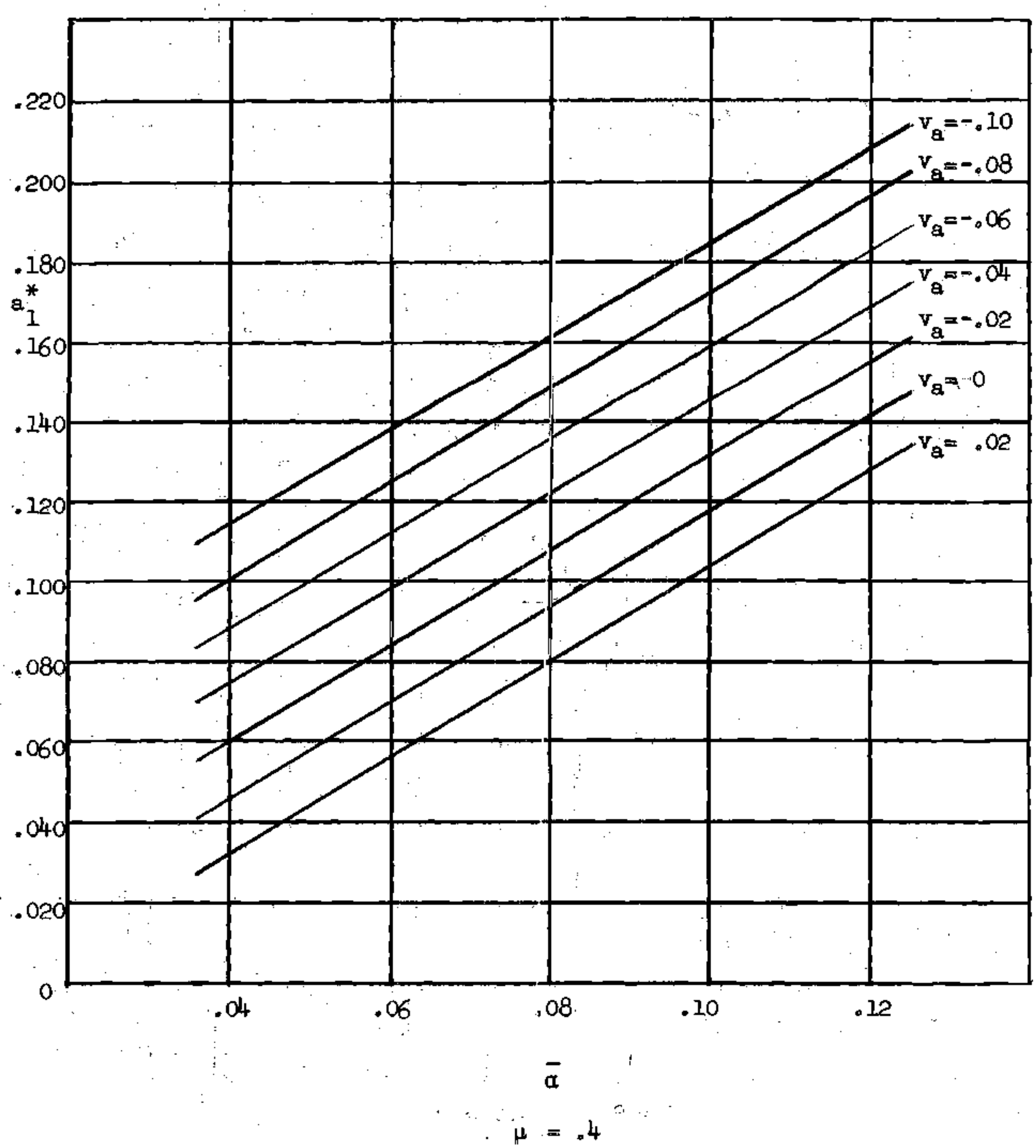
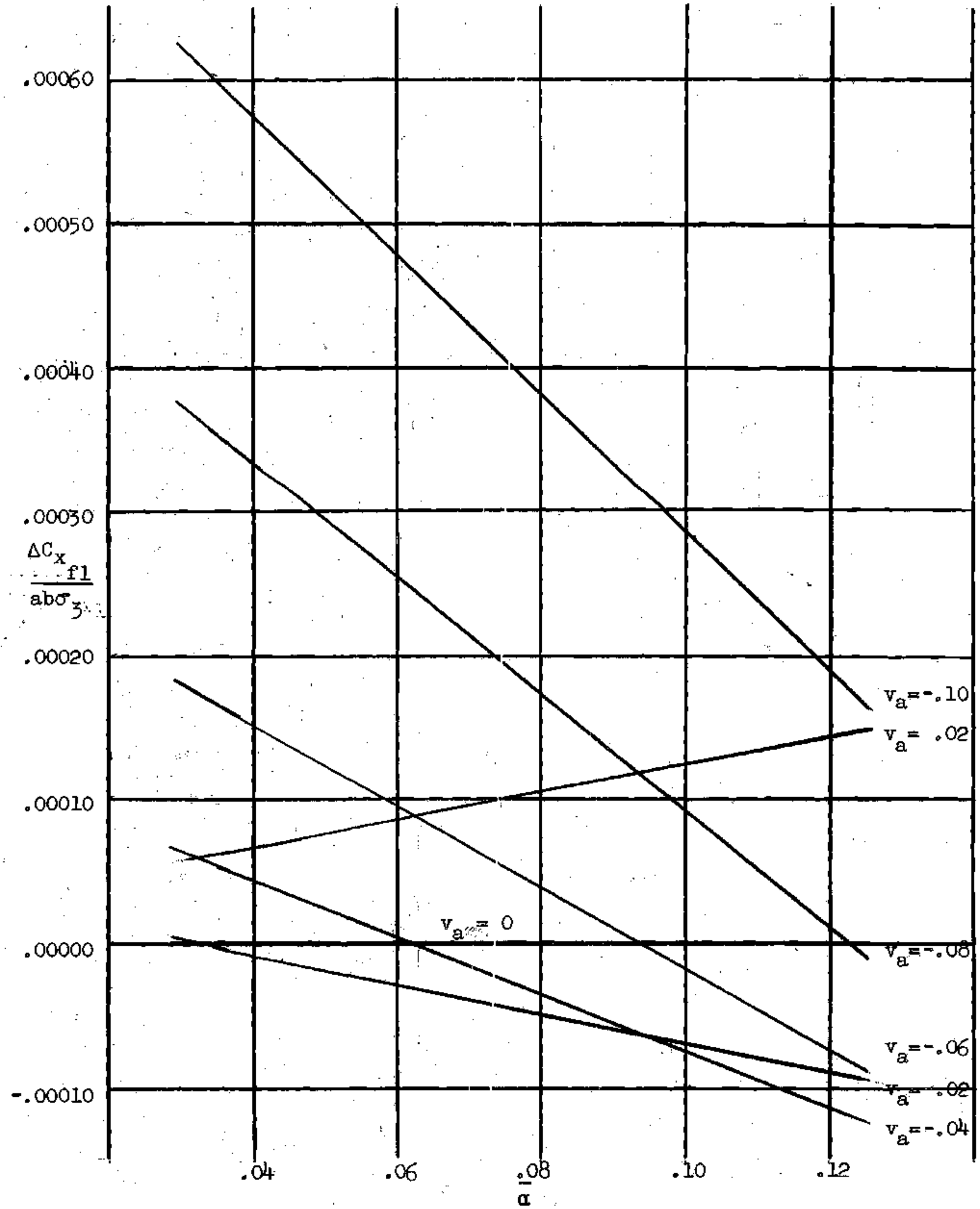


Figure 5(d). Reduced Lateral Cyclic Pitch.



$\mu = .1$

Figure 6(a). First Component of Fundamental Rotor X-Force.

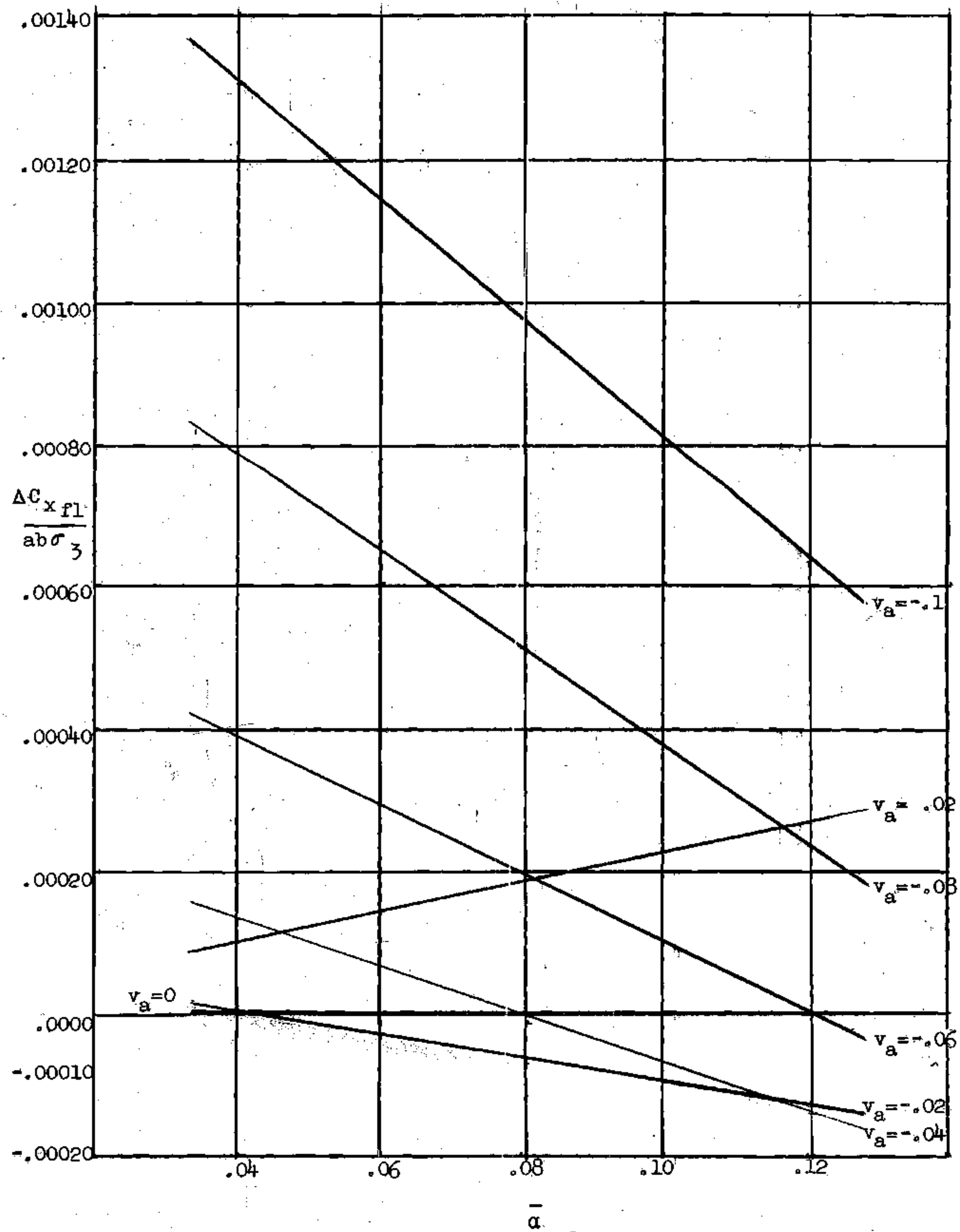


Figure 6(b). First Component of Fundamental Rotor X-Force.

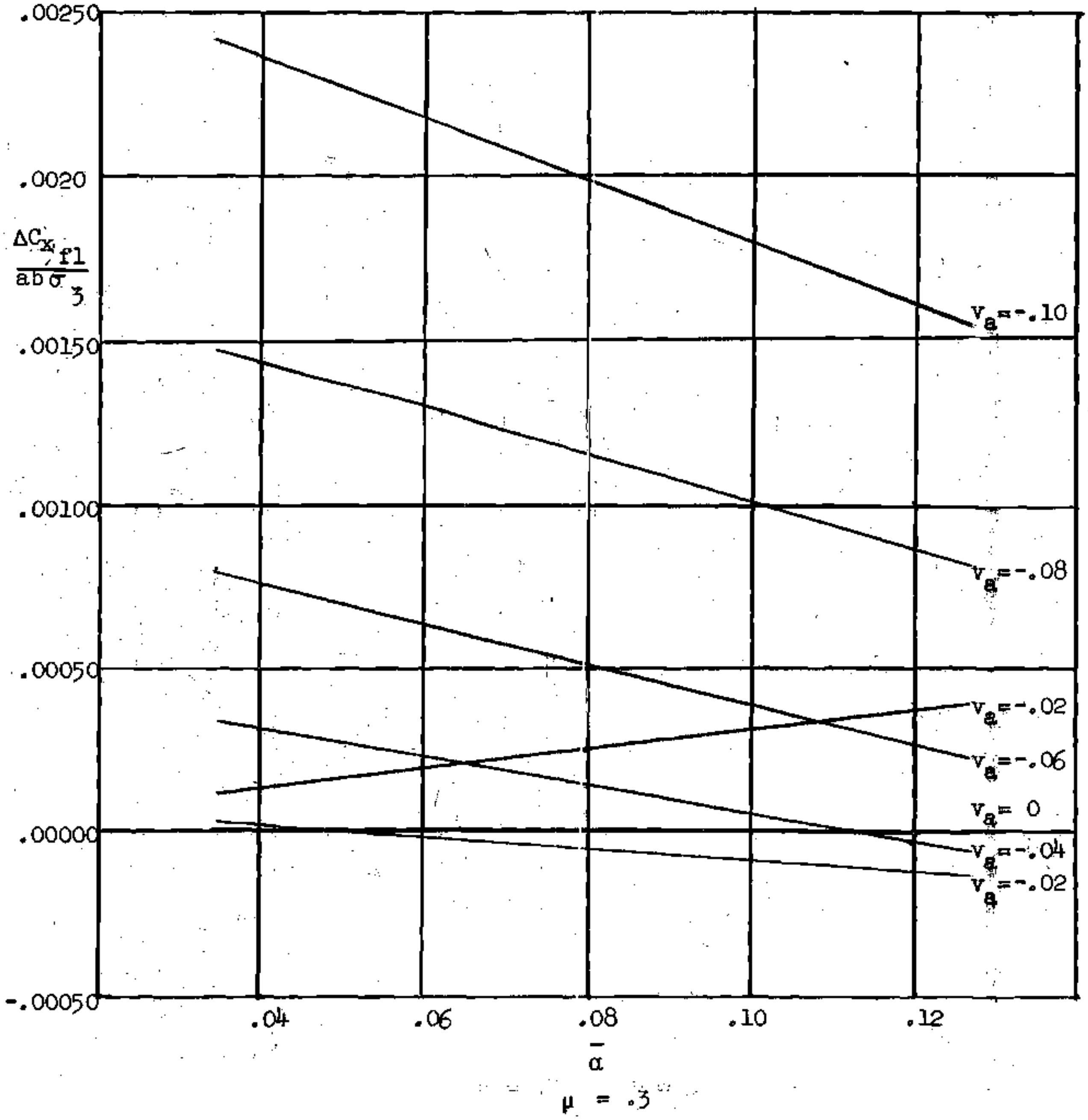
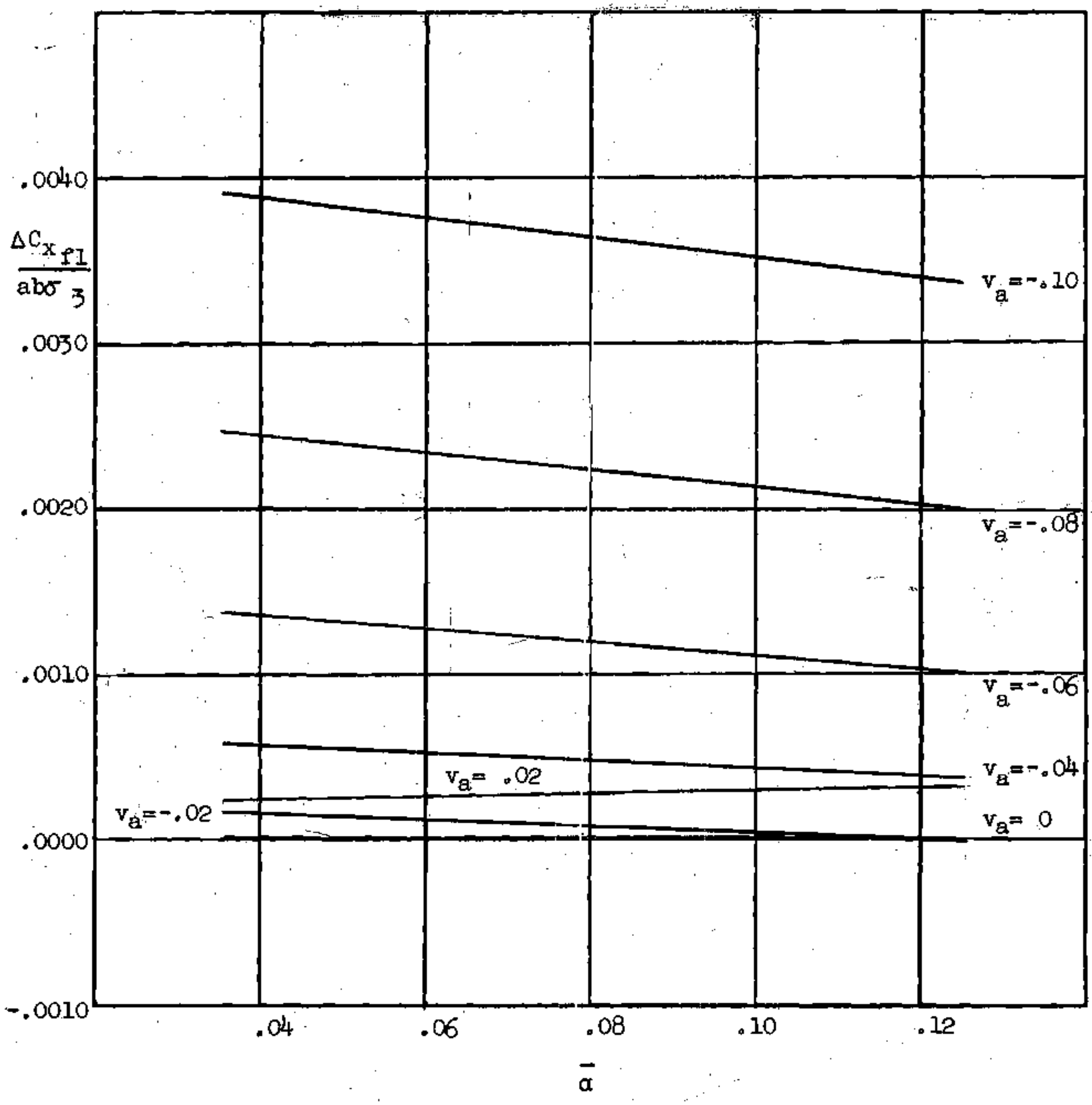
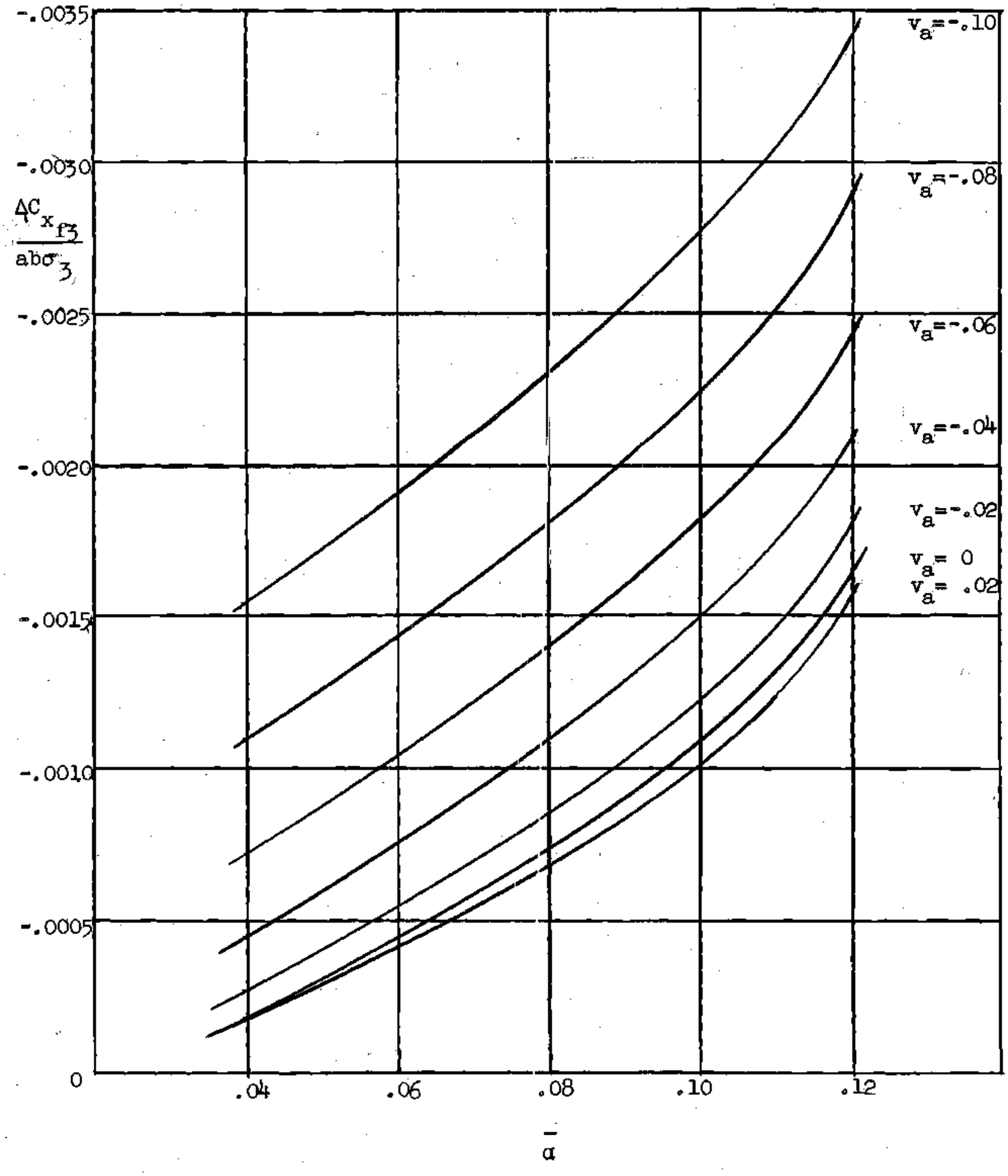


Figure 6(c). First Component of Fundamental Rotor X-Force.



$\mu = .4$

Figure 6(d). First Component of Fundamental Rotor X-Force.



$\mu = .1$

Figure 7(a). Second Component of Fundamental Rotor X-Force.

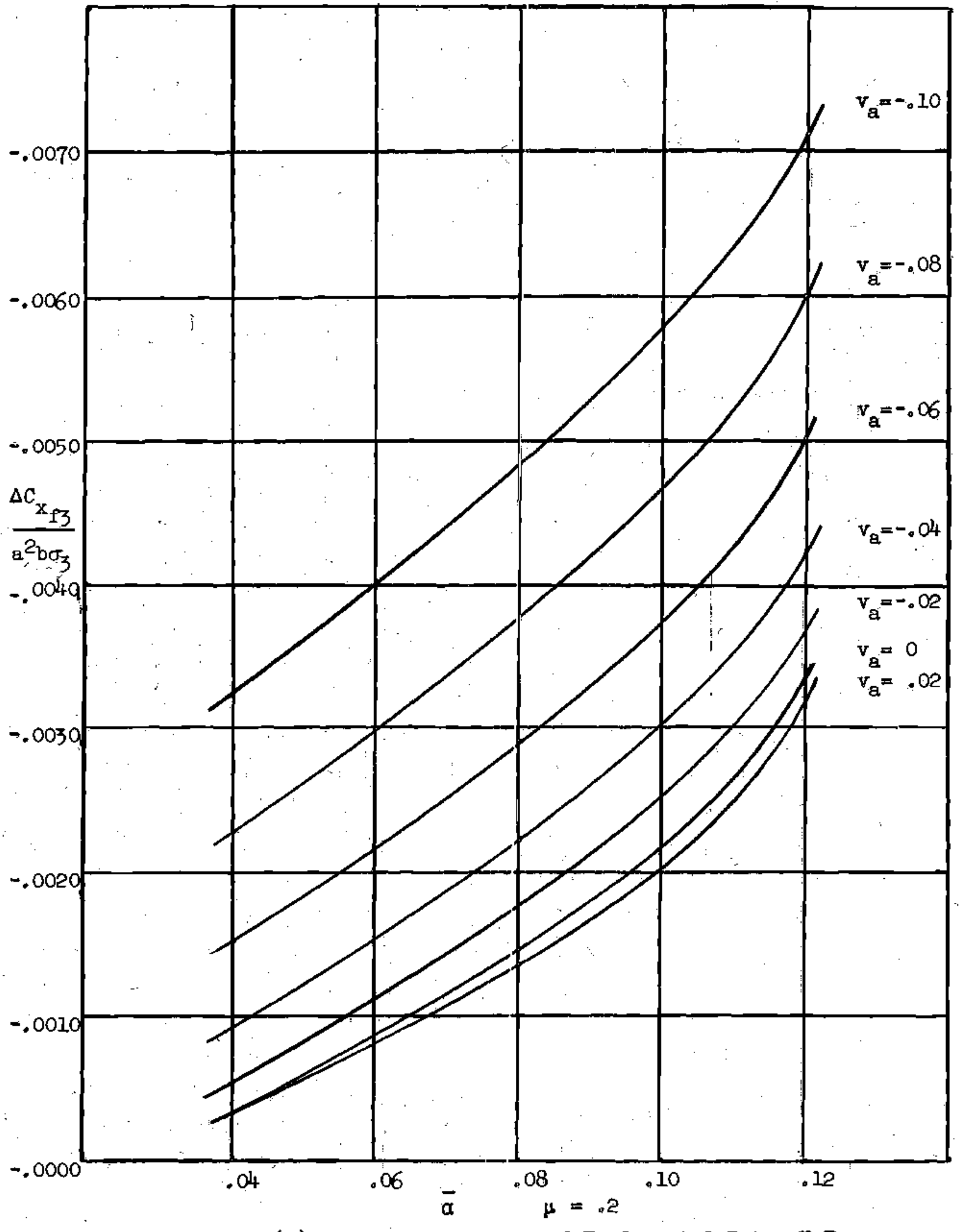


Figure 7(b). Second Component of Fundamental Rotor X-Force.

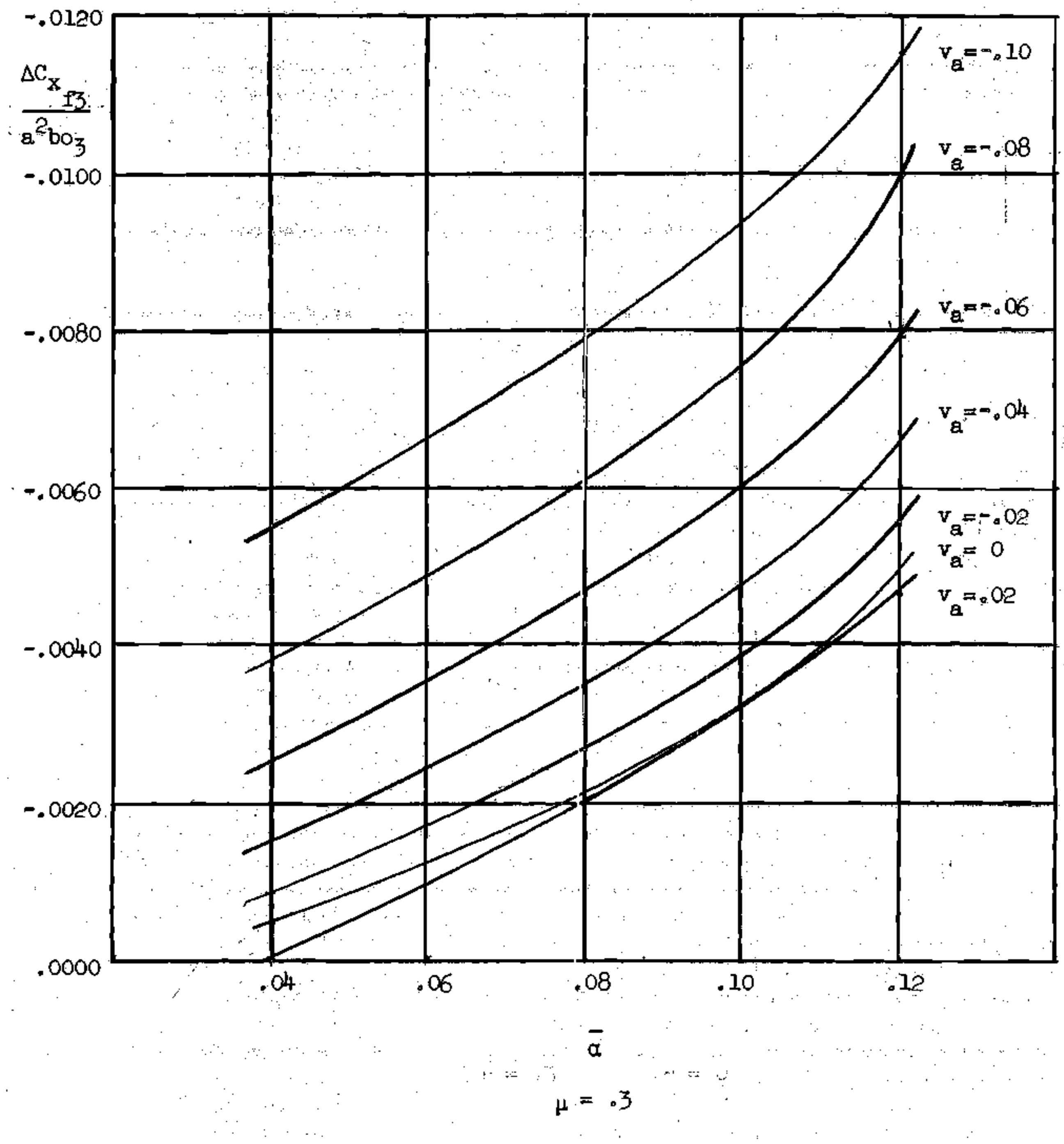


Figure 7(c). Second Component of Fundamental Rotor X-Force.

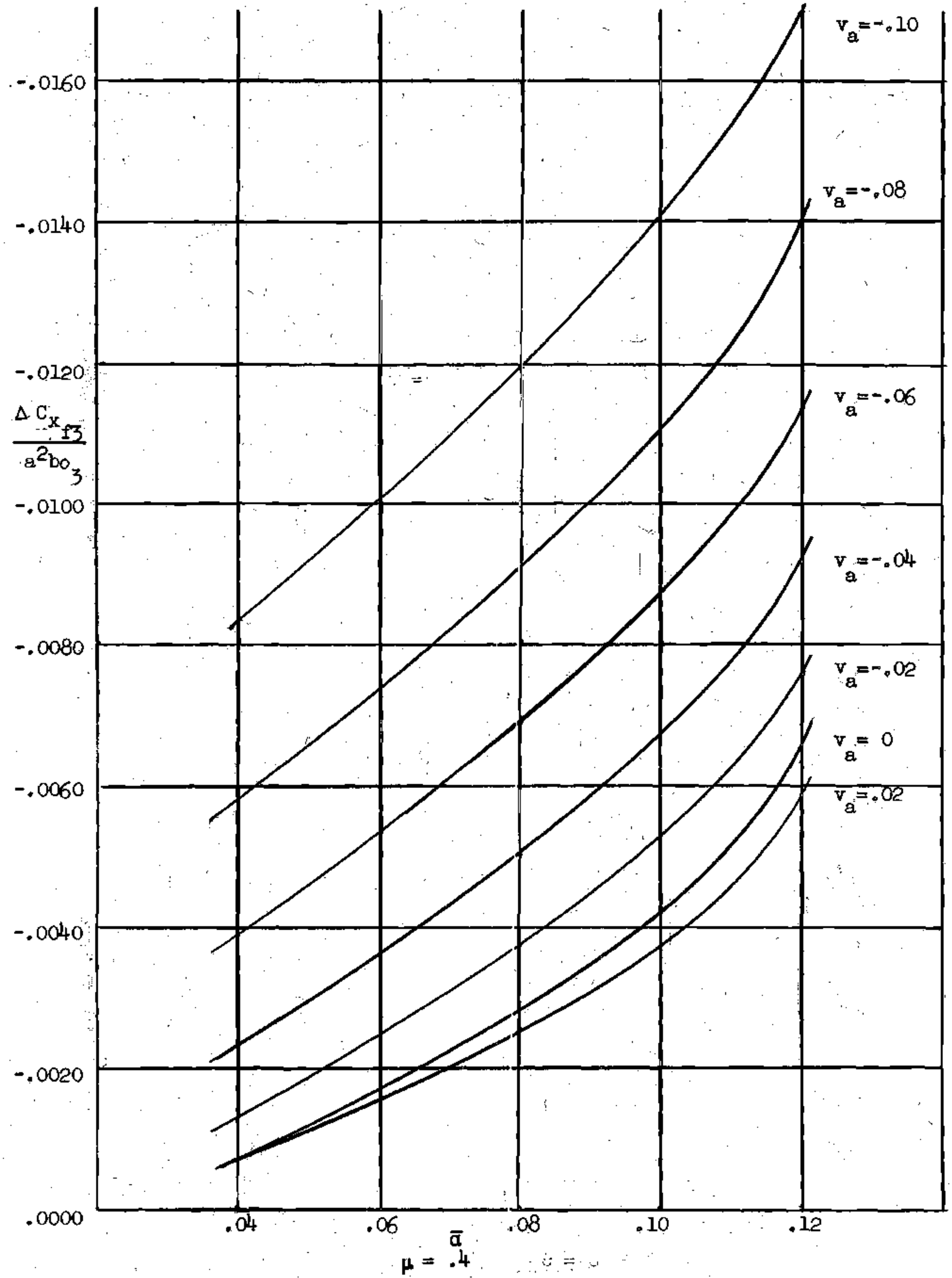


Figure 7(d). Second Component of Fundamental Rotor X-Force.

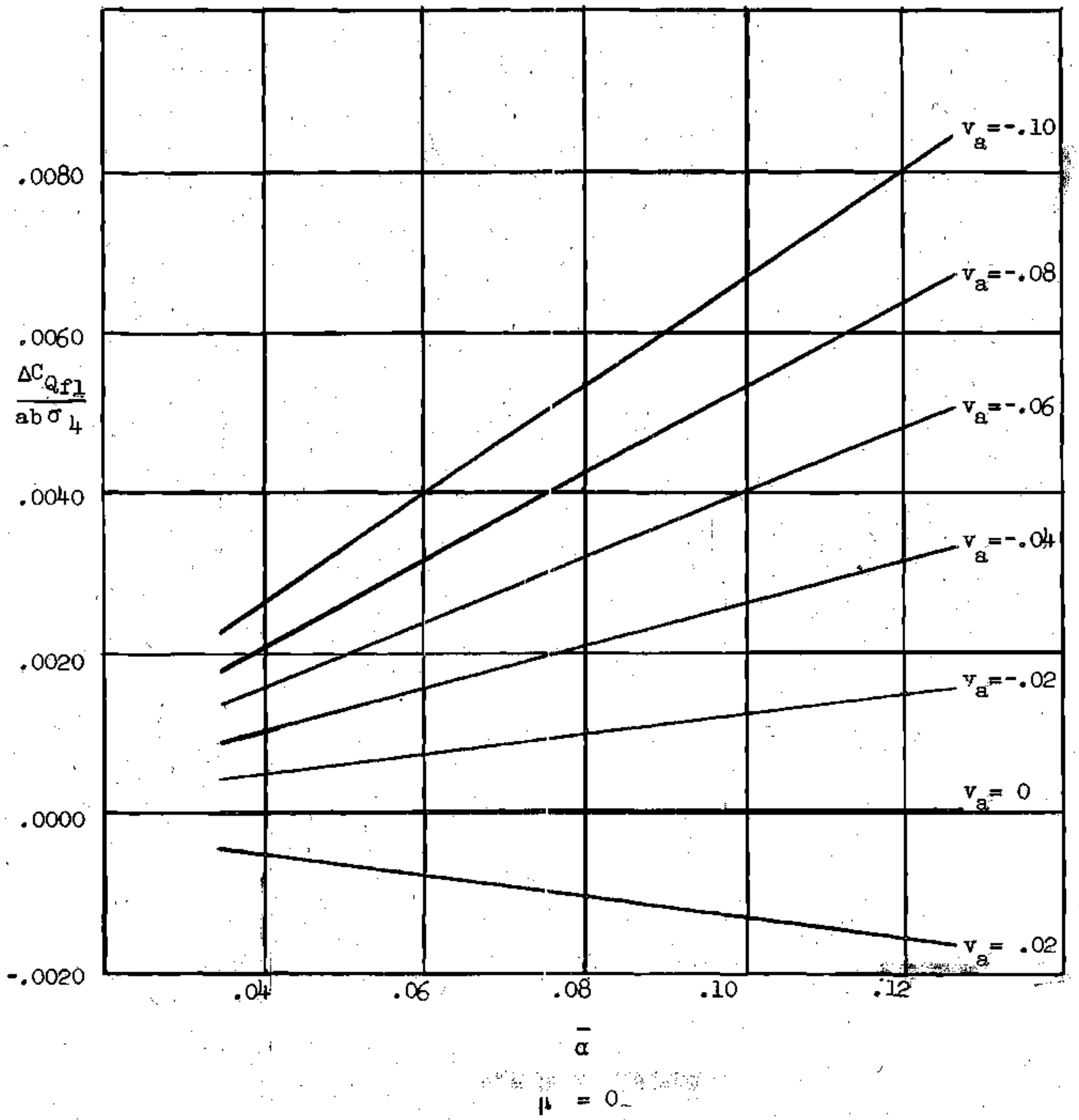


Figure 8(a). First Component of Fundamental Rotor Torque.

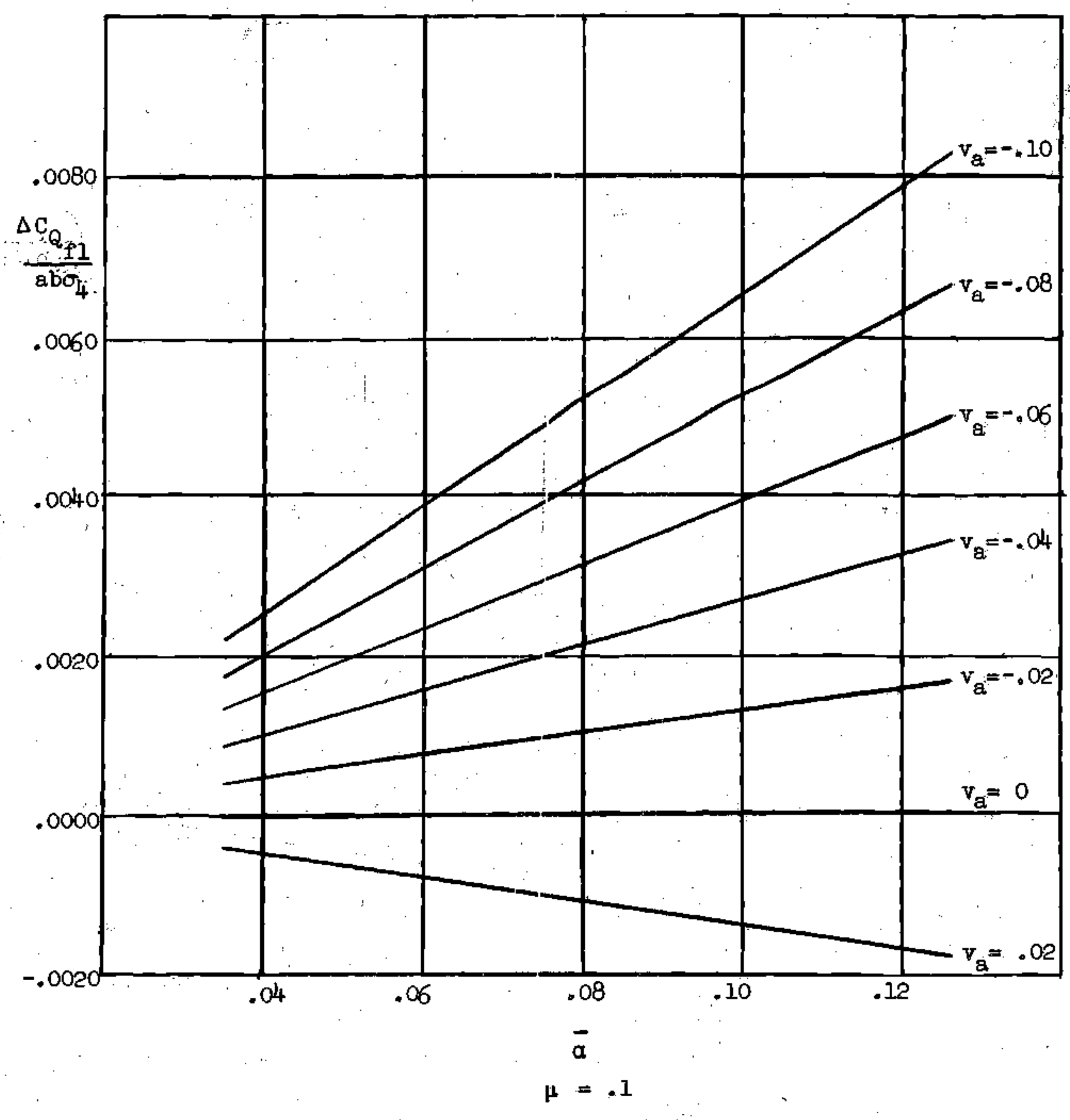


Figure 8(b). First Component of Fundamental Rotor Torque.

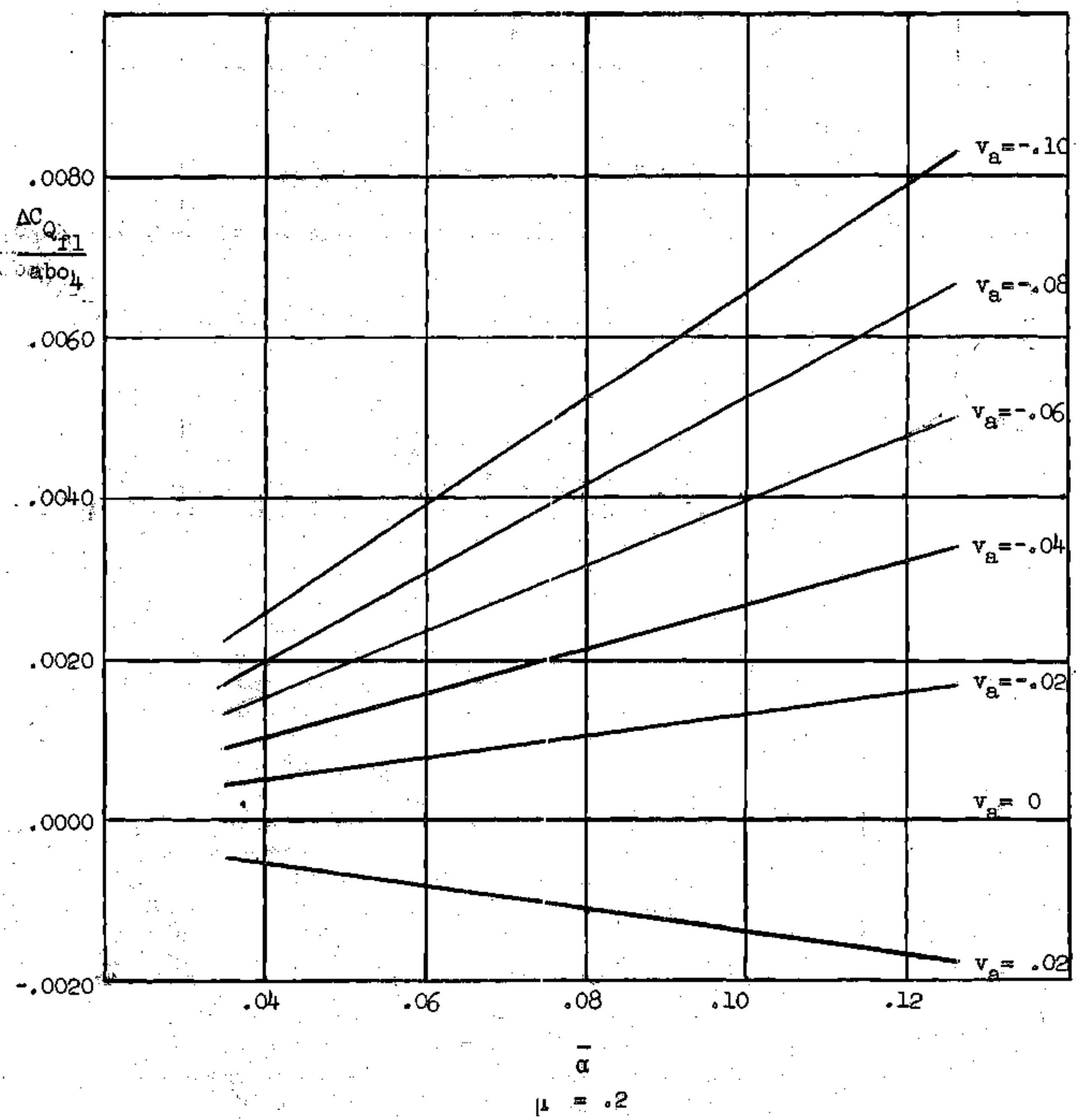


Figure 8(c). First Component of Fundamental Rotor Torque.

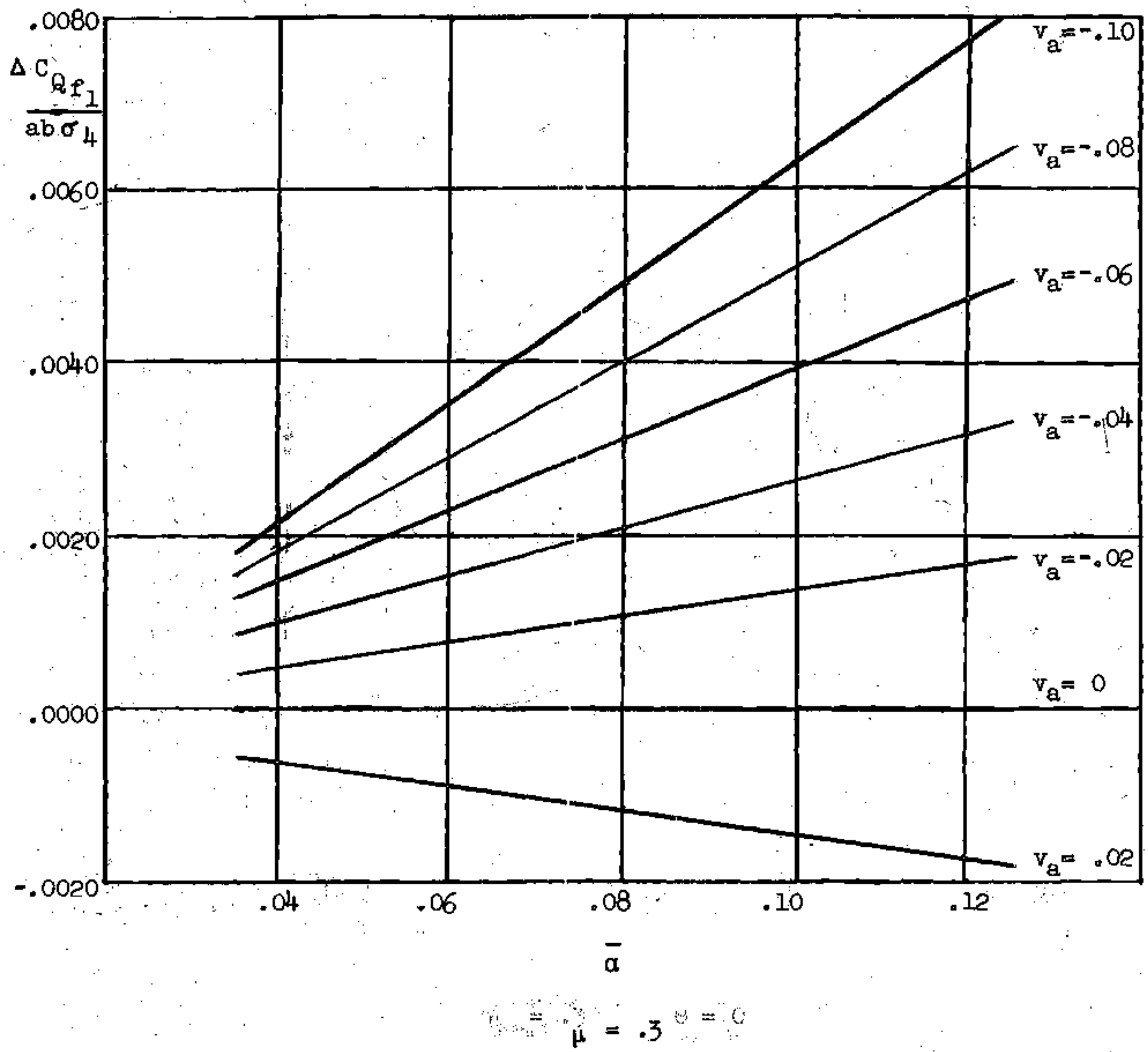


Figure 8(d). First Component of Fundamental Rotor Torque.

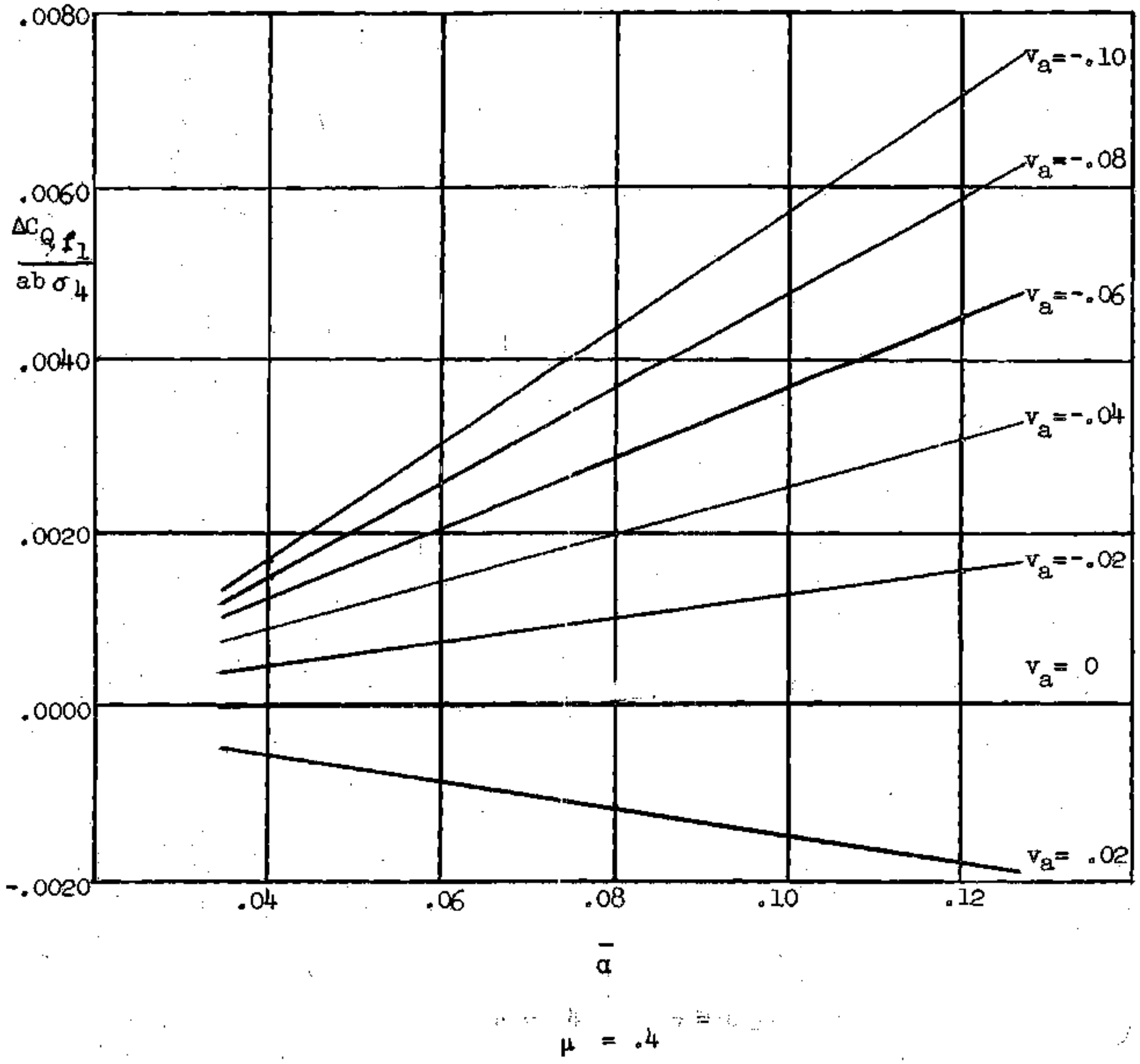


Figure 8(e). First Component of Fundamental Rotor Torque.

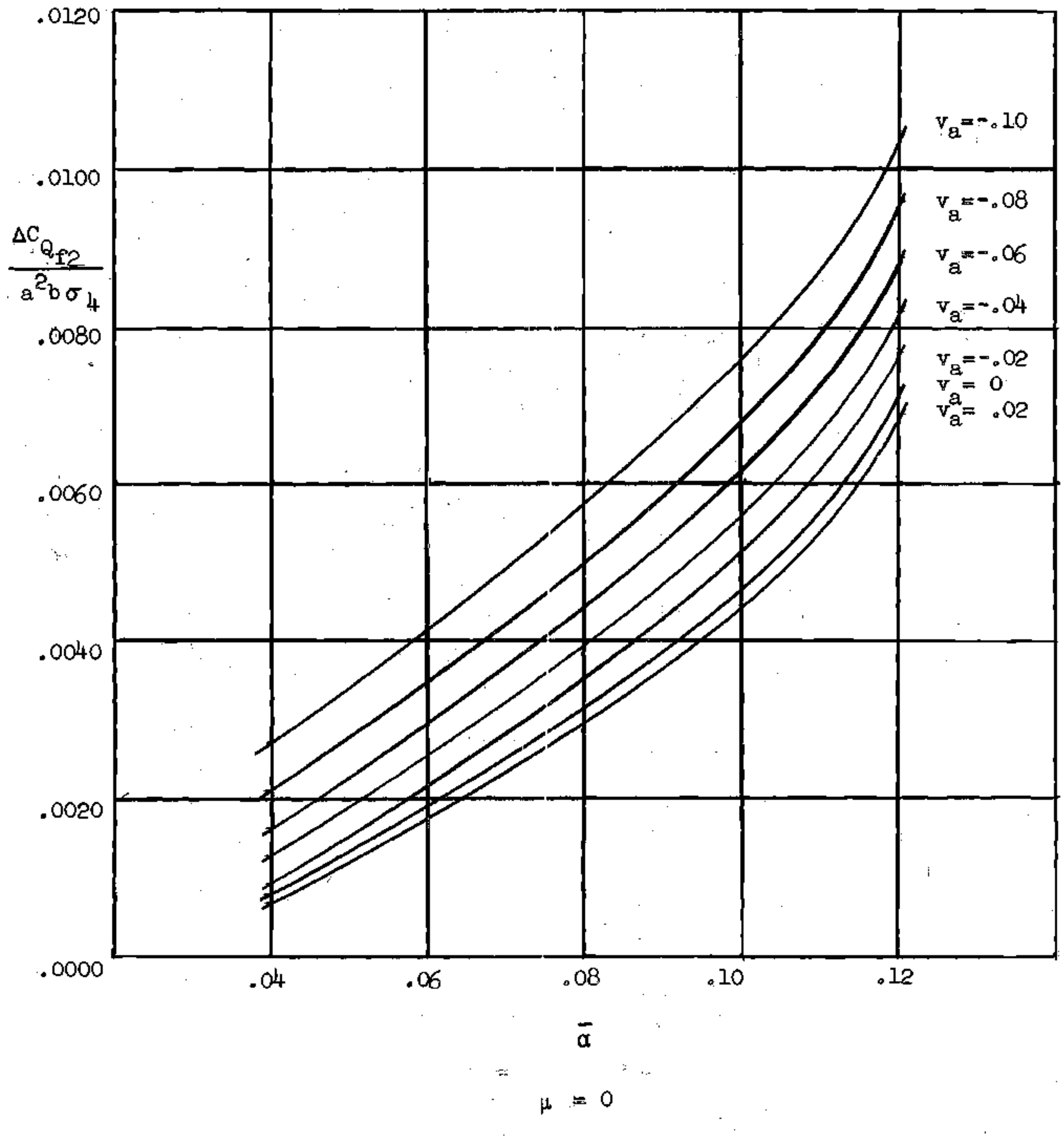


Figure 9(a). Second Component of Fundamental Rotor Torque.

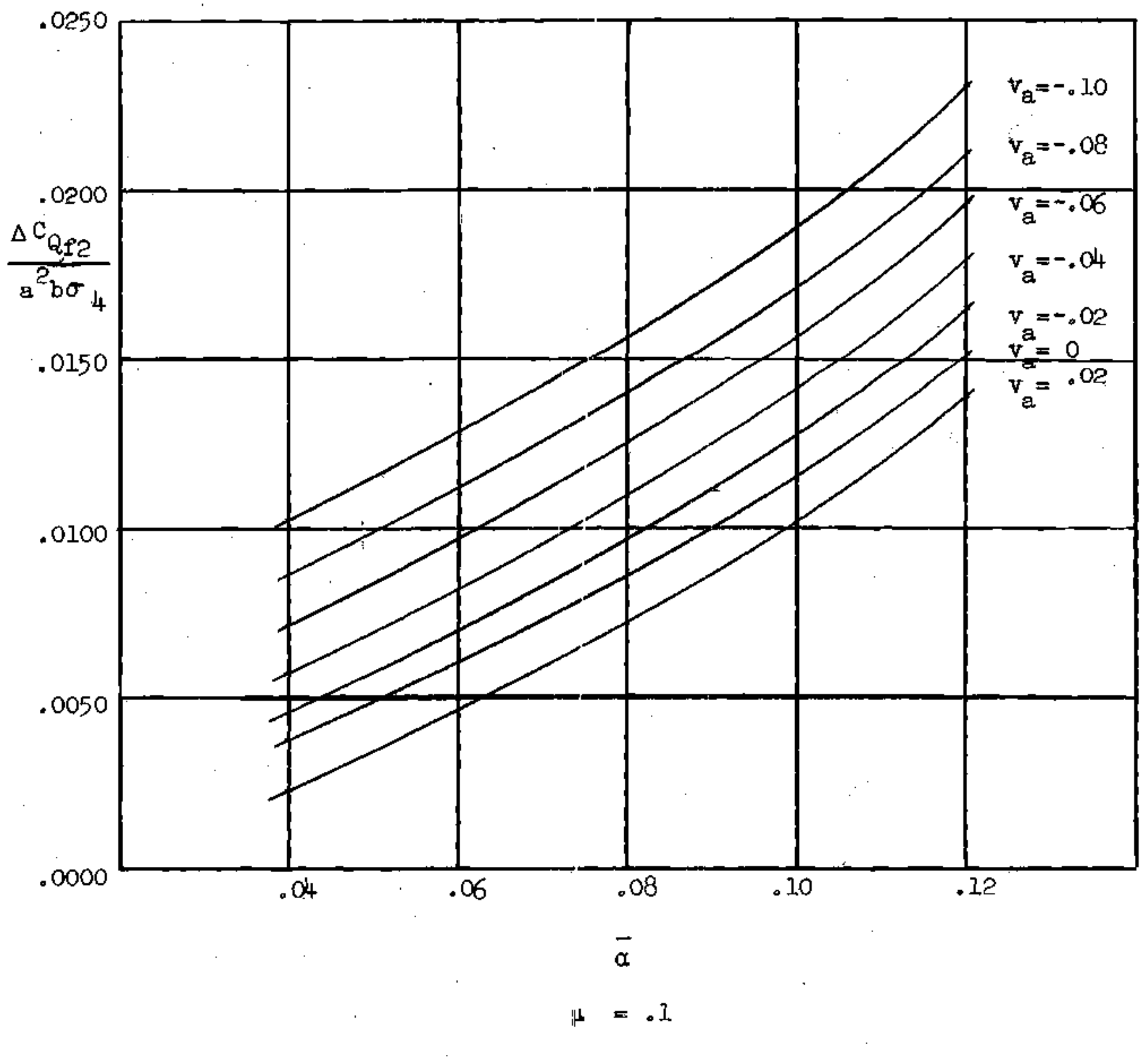


Figure 9(b). Second Component of Fundamental Rotor Torque.

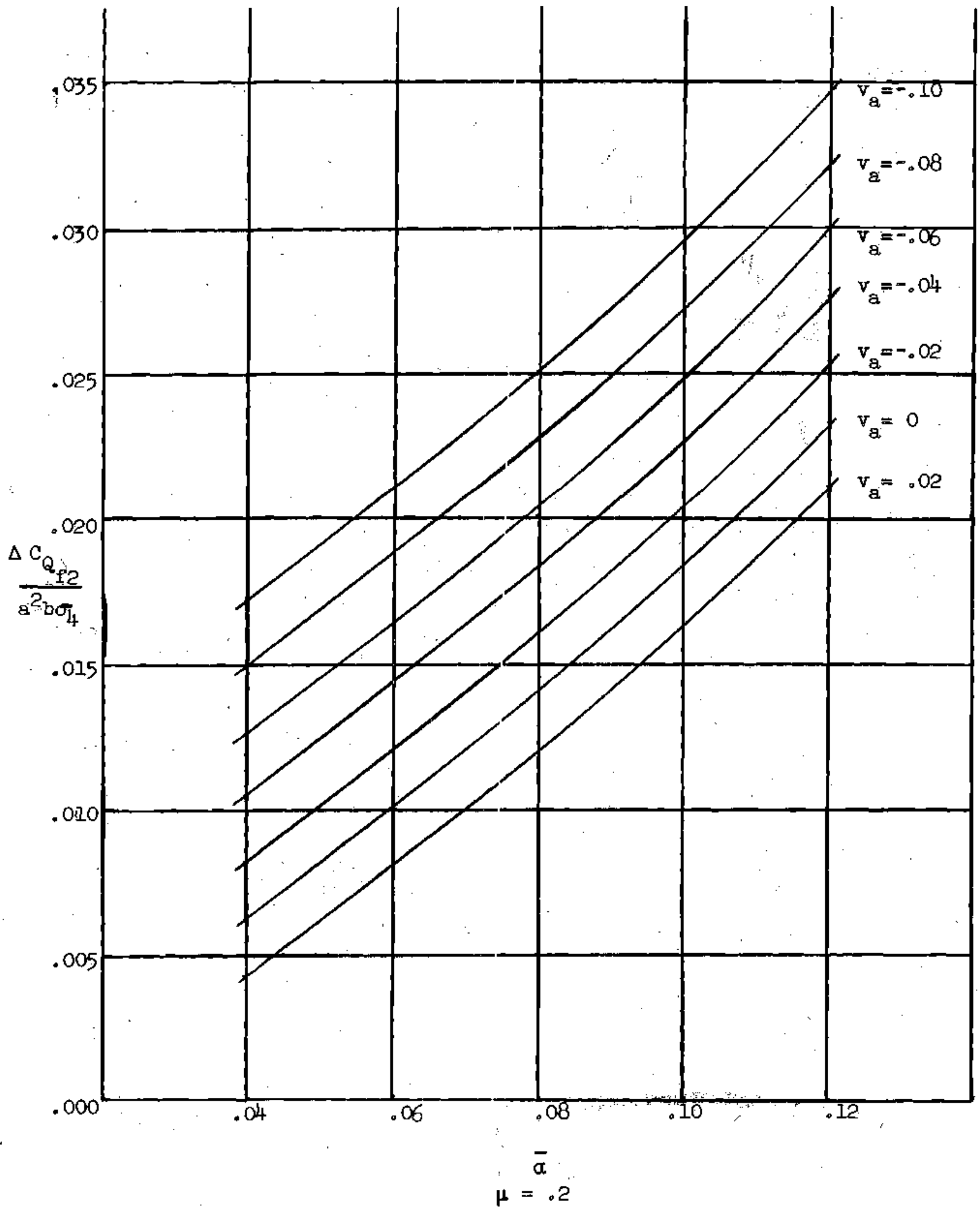


Figure 9(c). Second Component of Fundamental Rotor Torque.

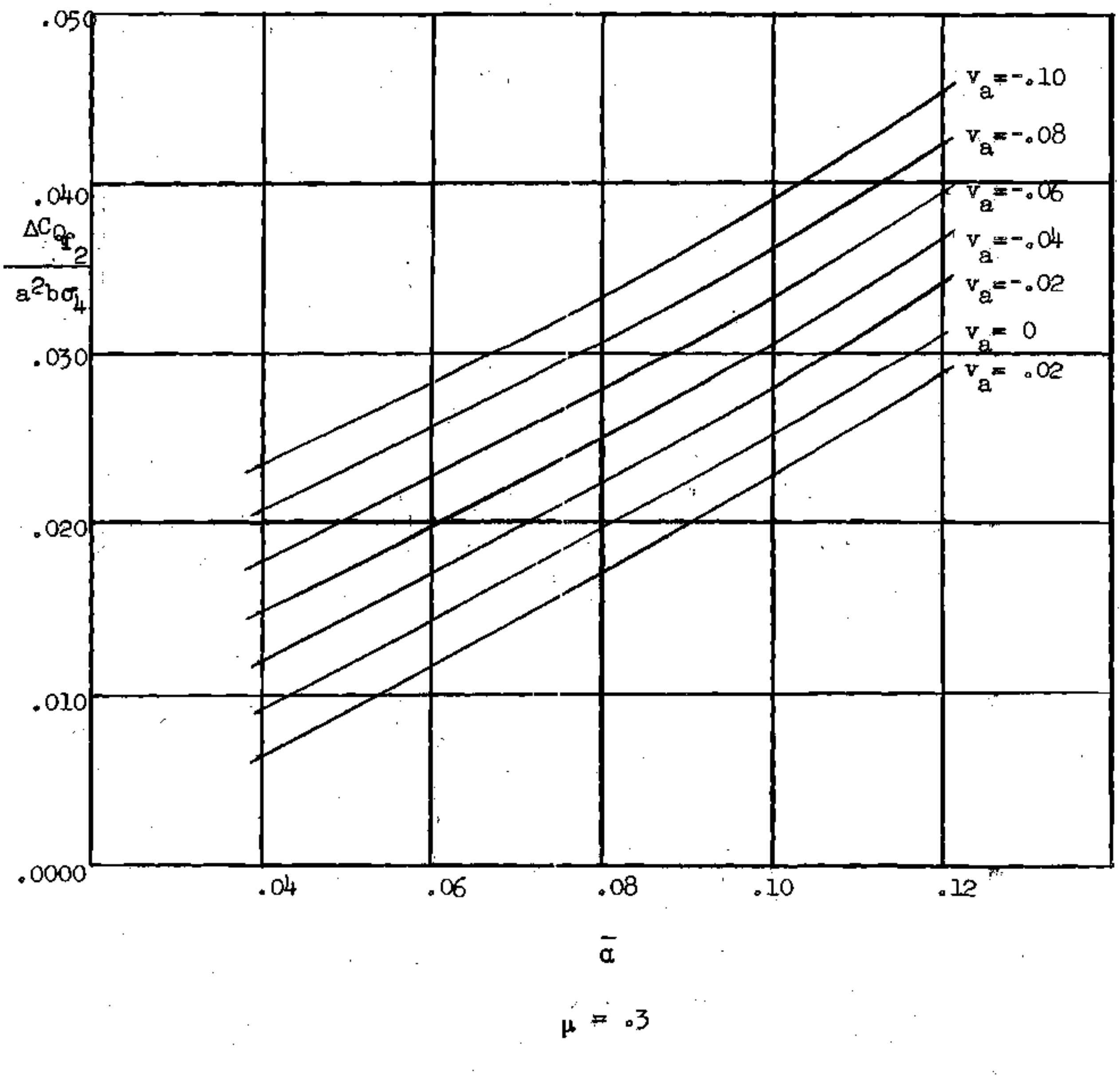


Figure 9(d). Second Component of Fundamental Rotor Torque.

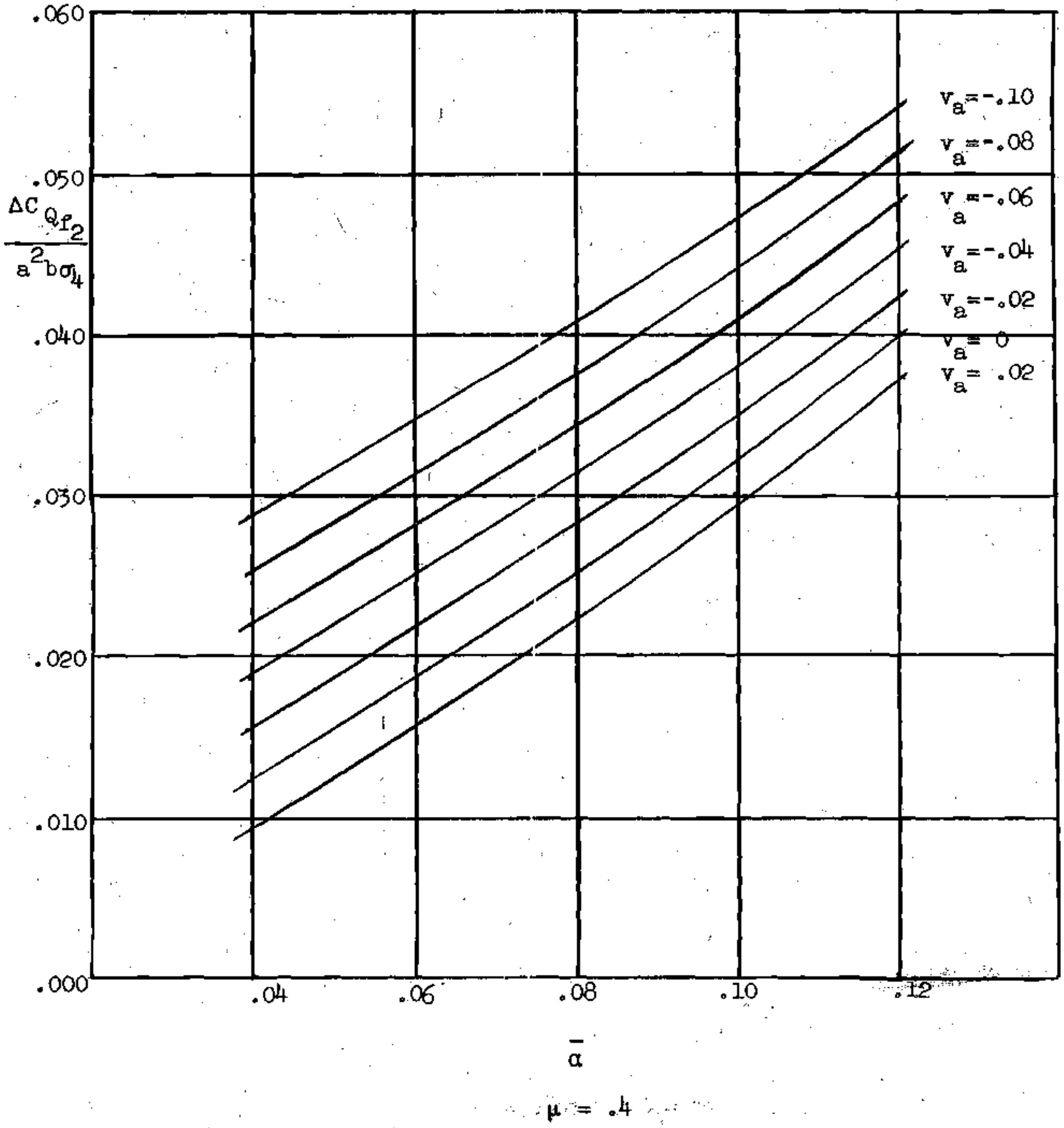
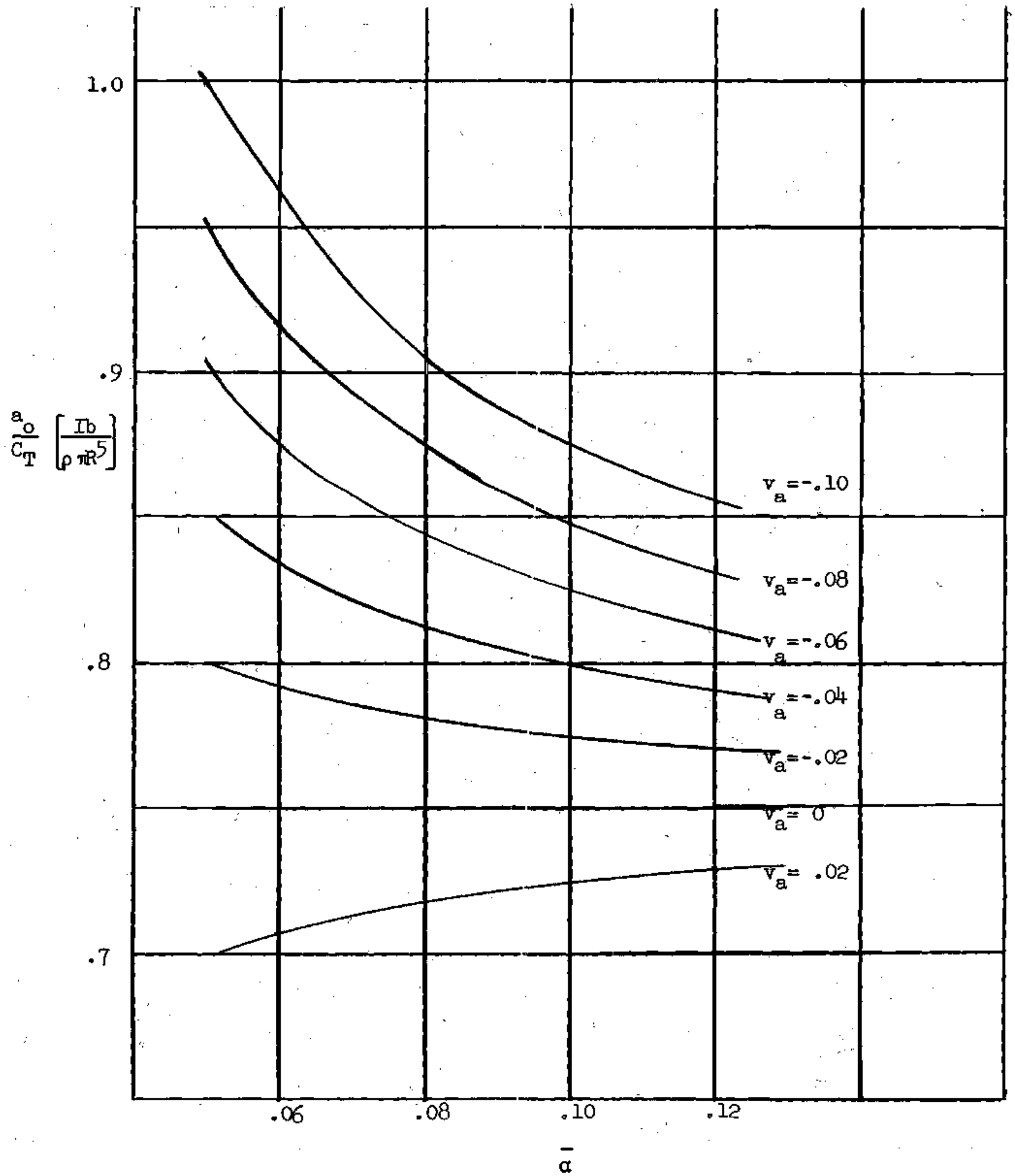


Figure 9(e). Second Component of Fundamental Rotor Torque.



$\mu = 0$

Figure 10(a). Coning Angle.

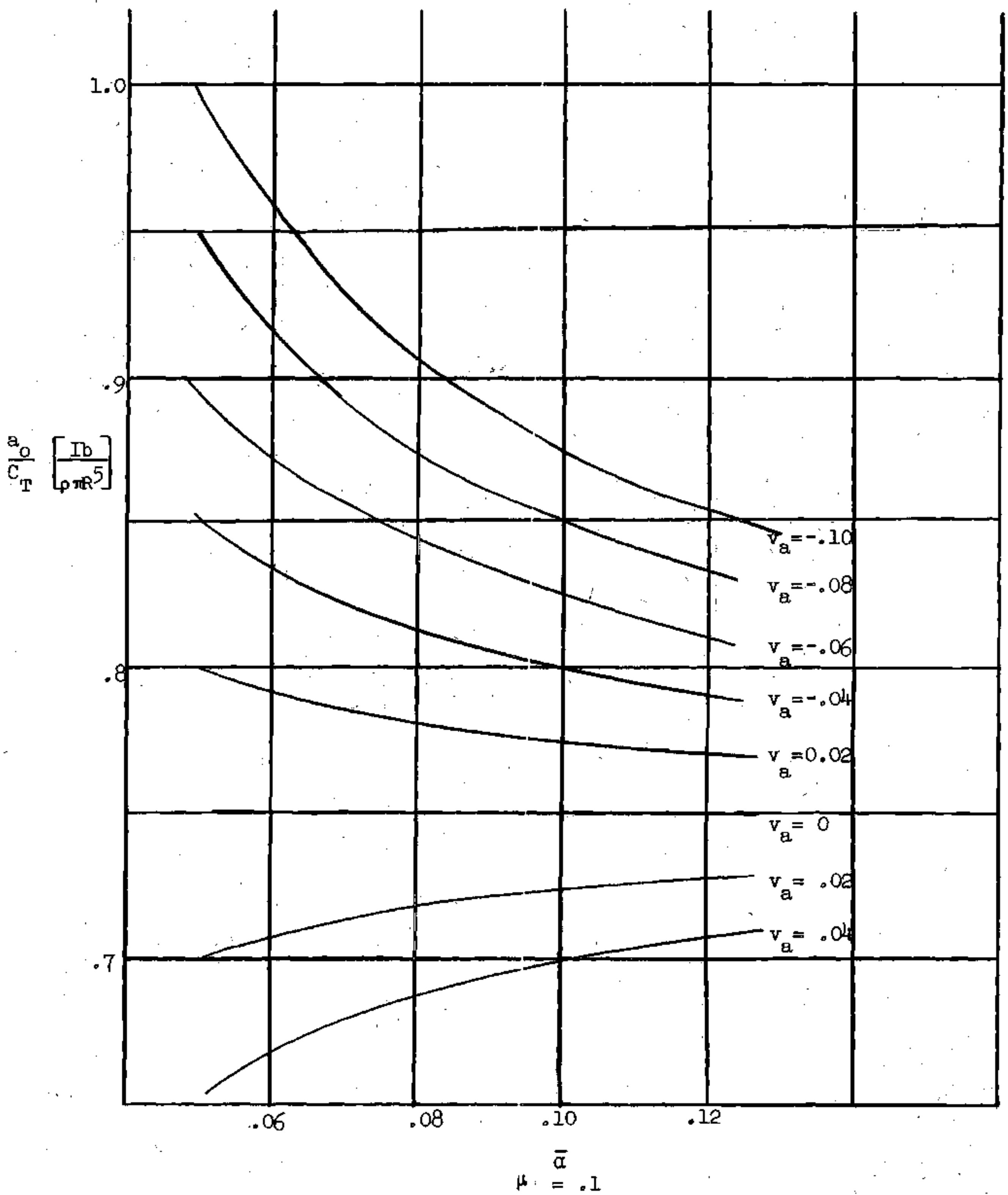


Figure 10(b). Coning Angle

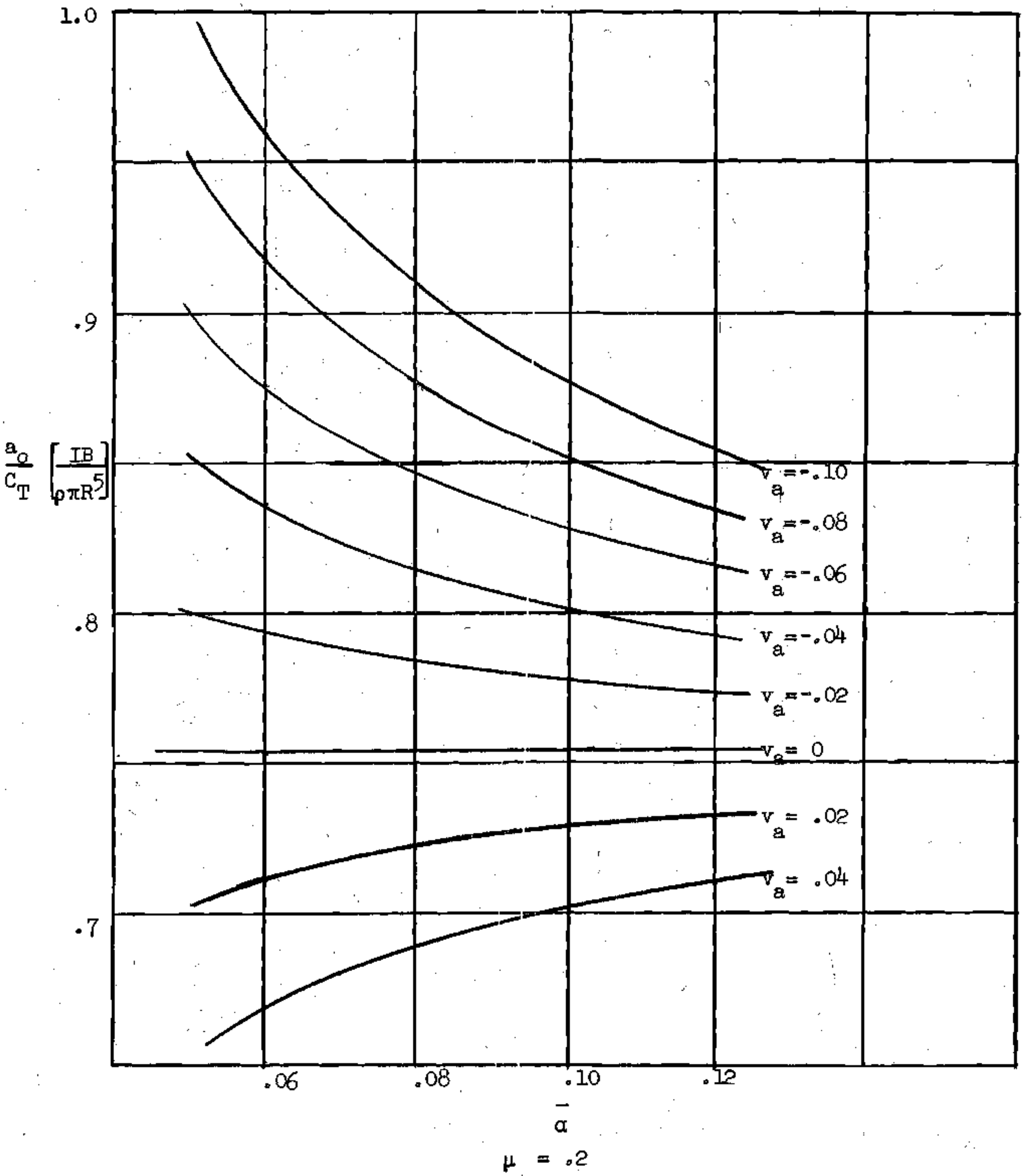


Figure 10(c). Coning Angle.

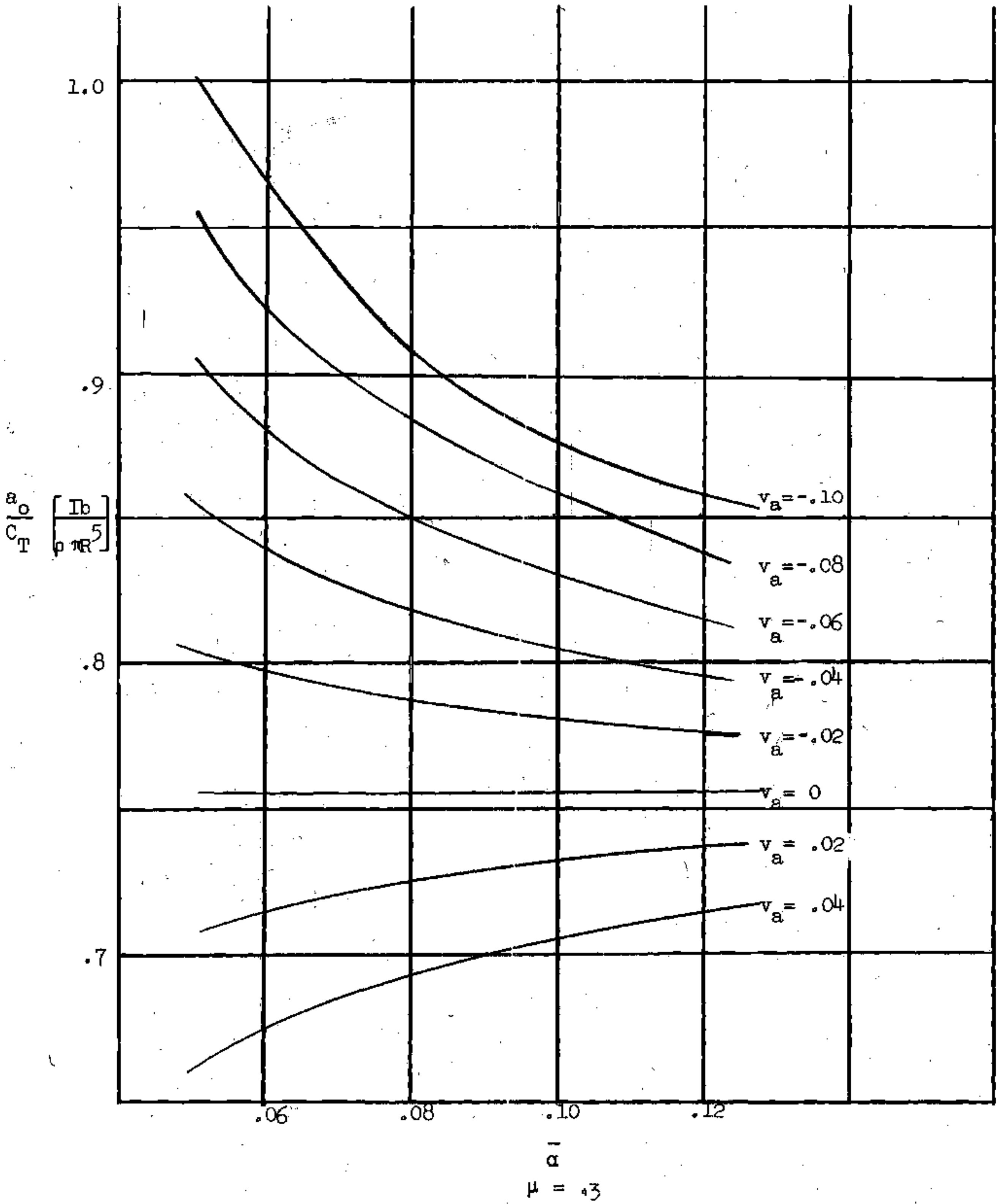


Figure 10(d). Coning Angle.

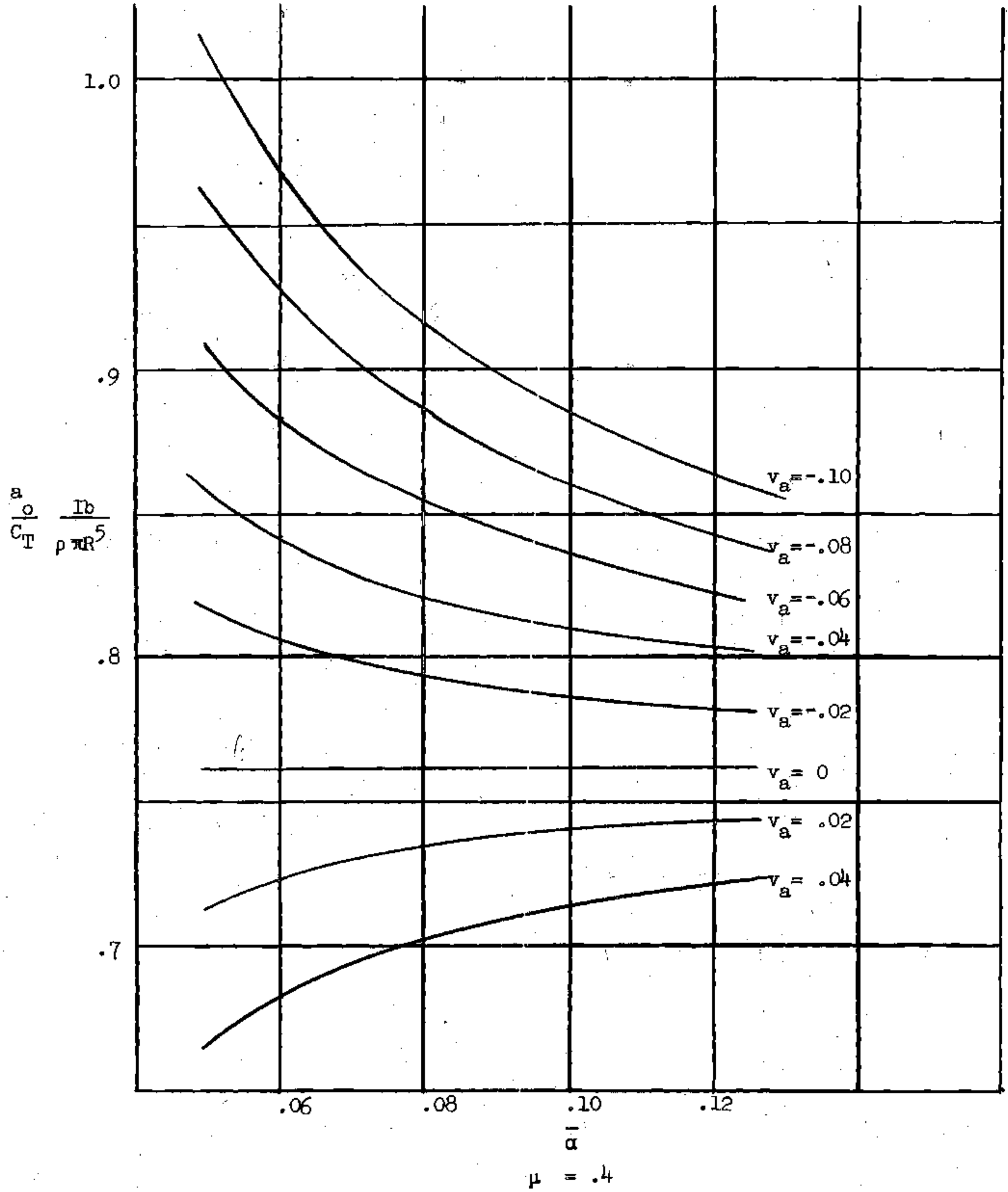


Figure 10(e). Coning Angle.

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